Derivation of a new viscous shallow water model with dependence on depth

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Considering the three-dimensional Navier-Stokes equations with free surface boundary condition in a domain with small depth, we study the derivation, using asymptotic analysis in the same way as in [1] and [2], of a new two-dimensional viscous shallow water model:

$$\frac{\partial h}{\partial t} + \nabla \cdot (h\vec{\mathbf{u}}) = 0 \tag{1}$$

$$\frac{\partial \vec{\mathbf{u}}}{\partial t} + \nabla\vec{\mathbf{u}} \cdot \vec{\mathbf{u}} - \vec{\mathbf{F}}_D + g\nabla h = -\frac{1}{\rho_0}\nabla p_s - g\nabla H + \vec{\mathbf{F}}_A \tag{2}$$

$$\frac{\partial \vec{\gamma}^j}{\partial t} + \nabla\vec{\gamma}^j \cdot \vec{\mathbf{u}} - (\nabla\vec{\mathbf{u}})^T \cdot \vec{\gamma}^j = \vec{\mathbf{F}}_V^j \quad (j = 0, 1, \ldots, N) \tag{3}$$

$$u = \bar{u} + h\sum_{j=0}^{N}\left[\frac{\gamma_2^j}{j+1}\left(\left(\frac{z-H}{h}\right)^{j+1} - \frac{1}{j+2}\right)\right] \tag{4}$$

$$v = \bar{v} - h\sum_{j=0}^{N}\left[\frac{\gamma_1^j}{j+1}\left(\left(\frac{z-H}{h}\right)^{j+1} - \frac{1}{j+2}\right)\right] \tag{5}$$

$$\vec{\mathbf{F}}_D = \nu\left\{\Delta\vec{\mathbf{u}} + \nabla(\nabla \cdot \vec{\mathbf{u}}) + \frac{1}{h}\left[(\nabla\vec{\mathbf{u}})^T + (\nabla\vec{\mathbf{u}})\right]\nabla h\right\} \tag{6}$$

where $\vec{\mathbf{u}} = (u, v)$ is the horizontal velocity, $\vec{\bar{\mathbf{u}}} = (\bar{u}, \bar{v})$ is the depth-averaged horizontal velocity, $h(t, x, y)$ describes the total length of the water column located at the (x, y) coordinate, $z = H$ represents the description of the topography variations (supposed known), p_s is the pressure at the surface, $\vec{\mathbf{F}}_A$ are the applied forces (it typically includes the Coriolis force, the wind at the free surface and the friction at the bottom), g is the gravitational acceleration, ρ_0 the density, ν the horizontal kinematic viscosity, and $\vec{\mathbf{F}}_V^j$ are known explicit functions of γ_1^j and γ_2^j.

Expressions (4)-(5) give us the horizontal velocity for all z and not only the depth-averaged horizontal velocity. There are two major novelties in the new shallow water model that we have obtained: the new diffusion terms and the dependence on depth of the velocities.

References

[1] José M. Rodríguez and R. Taboada-Vázquez. “A new shallow water model with explicit polynomial dependence on depth”. *Proceedings ECCOMAS 2008*, Venice (Italy), 2008.

[2] José M. Rodríguez and R. Taboada-Vázquez. “Bidimensional shallow water model with polynomial dependence on depth through vorticity". *Journal of Mathematical Analysis and Applications*, **359** (2), pp. 556–569 (2009).