Equalizers in Mobile Communications

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Modern cellular wireless systems would not be possible without high-performance equalizers. Only through them signal impairments introduced by the wireless channel can be compensated for. This tutorial serves as an introduction to the challenging design of equalizers. By introducing a general framework based on the Bezout Theorem we show that the most sophisticated equalizer concepts can be derived, thus allowing finite-length solutions to be found. Based on a canonical receiver structure, we also show how equalizers improve the Signal-to-Interference and Noise Ratio (SINR) under different sources of interference such as temporal, spatial, code and multiuser. We furthermore address the challenges of adaptive equalizers as they are currently typical solutions for low-cost devices. Although research on equalizers has been performed for many decades, we finish the tutorial with some open issues that are still keeping the field alive.

Motivation

Everybody in the modern world has some idea about what an equalizer is. Such devices exist in almost all HiFi systems and as their name associates, they allow for the equalization or for the balancing of sound. Low-frequency woofers or high-frequency tweeters can be turned up while middle frequencies can be lowered in order to flatten the overall spectrum, for example, to compensate shortcomings of the loudspeakers or the room acoustics; thus the word equalizer. In the wireless world however, the corresponding and at the same time equivalent relation in the time domain is more useful

$$H(\omega)=c \Leftrightarrow h(t)=c\delta(t). \qquad (1)$$

If an equalized channel $H(\omega)$ is constant over frequency ω (flat spectrum or frequency-flat channel), then its impulse response $h(t)$ over time t is very sharp (i.e., a Dirac delta distribution). For data symbols over a wireless (or cable) channel this means absence of inter-symbol interference as data symbols are not overlapping in time. Equalizers are nowadays not only found in HiFi audio and wireless cellular systems but also in control systems like reading high-speed magnetic discs, satellite communications, car-to-car communications and many more.

A Brief History of Equalizers: from Bob Lucky to Syed Jafar

Most researchers recall the famous papers of R.W. Lucky [4, 5] in 1965 and in 1966, published in the Bell System Technical Journal, as the first milestone paper of adaptive equalization. Lucky provided a working algorithm suited for digital implementation and also coined the expression “Zero Forcing” (ZF) that has been used until today. But in fact, there had been even earlier work on adaptive equalizers. The earliest mentions of digital equalizers were probably made in 1961 [1, 2] and in 1964 [3]. Later, in 1967 a first substantial improvement was introduced, the Decision Feedback Equalizer (DFE) concept [6-8] that allowed finite-length solutions. While Lucky’s original idea was based on the ZF criterion, in 1969 the Minimum Mean Square Error (MMSE) equalizer was proposed [9] which offered considerably new insight and potential. A further milestone was introduced with the Fractionally Spaced Equalizer (FSE) in 1970 [10], which inspired a large amount of significant work [11–13]. In 1972 the Maximum Likelihood (ML) principle [14] was introduced. In the eighties, overview papers became popular [15, 16] and the major driving forces in research led to blind equalization methods, introduced in 1975 [17, 18]. A first overview paper on blind methods was [19], although the breakthrough in this field came later in 1991 [20, 21]. Prominent articles are [22–24]. Extensions for Multiple-Input Multiple-Output (MIMO) transmissions started with [25] in 1992, followed by many others, of which we cite just a few selected papers [26–30]. More modern overview papers are provided in [31–35].

Recent development also includes a filter operation at the transmitter. This so-called precoding operation facilitates the work of the equalizers in a multi-user scenario. Cadambe and Jafar showed in 2008 that under some circumstances, a clever precoding technique called Interference Alignment (IA) makes it possible to offer everybody in a MIMO cellular network half the degrees of freedom without interference [36].

Causing all the Trouble: The Wireless Channel

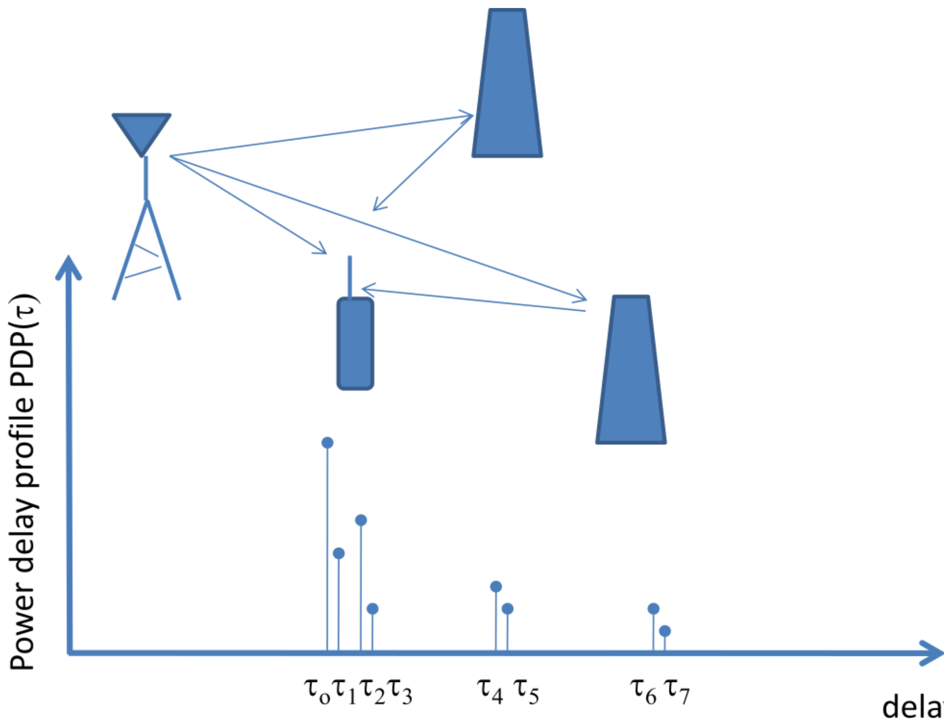

Figure 1: Power Delay Profile (PDP) of a wideband wireless channel. A multitude of reflections at different propagation delays generates the PDP.

The wireless transmission channel distorts data signals in two very different ways:

1) The propagation phenomenon that results in signals reaching a radio receiving antenna by several different paths is called a **multipath transmission**, and the resulting channel is termed as multipath channel. If measured the energy of the individual signals from different paths over arrival time, the so-called Power Delay Profile (PDP(τ)) of a wireless channel is obtained (see Figure 1). A well known phenomenon in this context is ghost images in analog TV, which appear if an airplane is flying nearby. Through its metallic body, a second reflected transmission path occurs with a different propagation delay, superimposing the first transmission path. Due to multipath propagation, the wireless channel causes such ghosts permanently. Equivalent terms to describe this effect are temporal dispersion as impulses disperse (smear out) in time, **intersymbol-interference** channels, and frequency-selective channels as equivalent transfer functions exhibit large ups and downs when plotted over frequency. Here, equalizers can substantially improve the situation by restoring frequency gaps. In fact, modern high data rate cable connections Asymmetric Digital Subscriber Line (ADSL) suffer from similar problems, the multipath being caused by multiple reflections in the cable and connectors. As Figure 1 shows, after a certain basic delay τ_0, the first reflection (often but not necessarily the strongest) is received. The following reflections typically occur in clusters, i.e., several of them appear in close vicinity. Once a band-

limited receiver experiences such a channel, say of bandwidth B, each path appears differently due to the band-limitation effect:

$$H(\omega)=\frac{1}{B}\begin{cases} e^{-j\omega\tau} & ;|\omega|\le\frac{B}{2} \\ 0 & ;else \end{cases} \quad\Leftrightarrow\quad \mathrm{sinc}(t-\tau)=\frac{\sin\left(\pi B(t-\tau)\right)}{\pi B(t-\tau)} \tag{2}$$

Here, we assumed an ideal low-pass filter. Particular filters may cause different shapes, for example root-raised cosine forms are very common. Thus, rather than a single reflection at delay τ, the receiver observes a theoretically unlimited (double infinite) channel impulse response. Once the observed signals are sampled, let us say at multiples of T, an infinitely long sequence is obtained in general.

Figure 2 shows such an effect for the simple case of a two-path channel model in which the direct path is received with only 40% strength while the second path, with a delay of $\tau=T/2$ is received with full strength (see Equation (3)). In Equation (4) the corresponding sequence, sampled at kT is presented:

$$h(t)=0.4\delta\frac{\sin\left(\pi Bt\right)}{\pi Bt}+\frac{\sin\left(\pi B\left(t-\frac{T}{2}\right)\right)}{\pi B\left(t-\frac{T}{2}\right)}, \tag{3}$$

$$h_n=0.4\delta_n+\sum_{k=-\infty}^{\infty}\frac{\sin\left(\pi TB\left(k-\frac{1}{2}\right)\right)}{\pi TB\left(k-\frac{1}{2}\right)}\delta_{n-k}. \tag{4}$$

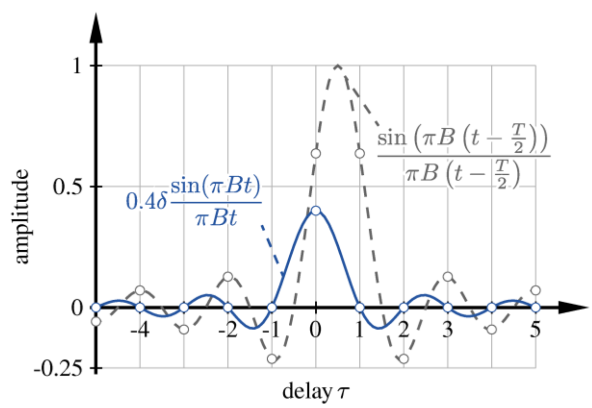

Figure 2: Channel impulse response: the two components of a two-path channel appear differently in band-limited form. Once sampled uniformly at kT, only the values marked by circles remain.

Based on such a simple example, we learn that even simple channel impulse responses, once they are band-limited and sampled, typically appear as double-infinitely long sequences. The mathematical model of such a transmission can be described by the following terms:

$$\mathbf{r}_k = \mathbf{H}\mathbf{s}_k + \mathbf{n}_k, \tag{5a}$$

$$\begin{bmatrix} r_k \\ r_{k-1} \\ \vdots \\ r_{k-R+1} \end{bmatrix} = \begin{bmatrix} h_0 & h_1 & \cdots & & \\ & h_0 & h_1 & \cdots & \\ & & \ddots & \ddots & \\ & & & h_0 & h_1 \end{bmatrix} \begin{bmatrix} s_k \\ s_{k-1} \\ s_{k-2} \\ \vdots \\ s_{k-S+1} \end{bmatrix} + \begin{bmatrix} n_k \\ n_{k-1} \\ \vdots \\ n_{k-R+1} \end{bmatrix}. \tag{5b}$$

We recognize the impulse response of the wireless channel in form of a rectangular-shaped Töplitz matrix $\boldsymbol{H}$. The observation vector $\mathbf{r}_k$ is typically additively disturbed by noise $\mathbf{n}_k$, disrupting the transmitted signal $\mathbf{s}_k$.

2) The **time-variant transmission effect** or **fading** causes individual transmission paths to alter their attenuation and even to disappear entirely at certain time instants (the received signal energy falls below the receiver sensitivity), originating so-called time-selective channels. The physical reason for this effect is destructive interference of electromagnetic waves by movement of transmitter, receiver, and/or scattering objects in the transmission path. If such movements are relatively slow (slow fading), adaptive equalizer algorithms may be able to catch up (tracking behavior) and compensate for them in a wide range. If a path entirely disappears, no equalizer can restore it. Such a fading effect typically occurs only for wireless channels and is absent (up to some very slow altering effects) in cable connections. As time variations typically occur jointly with frequency-selectivity of the channel (thus originating double-selective fading channels), an equalizer may have to deal with both effects at the same time.

   Modern wireless systems like WiMAX, UMTS LTE, and WiFi are based on OFDM transmissions, chopping a large frequency range into small bands. Inside such small bands, the channel can be described by a single complex-valued tap and thus exhibiting a frequency-flat response that can be easily compensated by the equalizer. However, under movement the neighboring bands can cause **inter-carrier interference** which is currently a big challenge for high-speed vehicles like fast trains.

Modern wireless systems may cause combinations of interference effects due to their sophisticated transmission modes. Once not only a single antenna for transmission and reception is employed but so-called MIMO systems, not only the channel capacity of a wireless system improves substantially but also its **spatial interference**. The corresponding transmission for a frequency-flat channel can be described as:

$$\mathbf{r}_k = \mathbf{H}\mathbf{s}_k + \mathbf{n}_k, \tag{6a}$$

$$\begin{bmatrix} r_k^1 \\ r_k^2 \\ \vdots \\ r_k^{N_R} \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & \cdots & \cdots & h_{1N_T} \\ h_{21} & h_{22} & h_{23} & \cdots & \vdots \\ \vdots & & \ddots & \ddots & \\ h_{N_R1} & \cdots & & h_{N_RN_T-1} & h_{N_RN_T} \end{bmatrix} \begin{bmatrix} s_k^1 \\ s_k^2 \\ \vdots \\ s_k^{N_T} \end{bmatrix} + \begin{bmatrix} n_k^1 \\ n_k^2 \\ \vdots \\ n_k^{N_R} \end{bmatrix}, \tag{6b}$$

which has a similar structure to Equation (5). However, the transmission matrix is no longer of Töplitz structure but of an arbitrary rectangular form, defined by the number of receive (N_R) and transmit (N_T) antennas. The noise vector $\mathbf{n}_k$ still consists of noise samples at the various N_R receive antennas, while the transmit vector $\mathbf{s}_k$ contains N_T data elements per time instant k. Due to the channel matrix $\mathbf{H}$, the transmit vector is spatially mixed.

Today, the most operated 3G wireless systems are based on Code Division Multiple Access (CDMA) signals, thus orthogonal codes are being employed in the downlink to distinguish the signals of each user. Unfortunately, once sent through a multipath channel, their orthogonality is lost and the codes are interfering with each other, causing **code interference**. Furthermore, neighboring base-stations may transmit at the same carrier frequency and cause **out-of-cell interference**. Such a transmission from M base-stations to a single user (or alternatively from M users to a single base-station) can be modeled as

$$\mathbf{r}_k = \sum_{m=1}^{M} \mathbf{H}_m \mathbf{s}_k^m + \mathbf{n}_k = \mathbf{H}\mathbf{s}_k + \mathbf{n}_k, \tag{7a}$$

$$\begin{bmatrix} r_k^1 \\ r_k^2 \\ \vdots \\ r_k^{N_R} \end{bmatrix} = \sum_{m=1}^{M} \begin{bmatrix} h_{11}^m & h_{12}^m & \cdots & \cdots & h_{1N_T}^m \\ h_{21}^m & h_{22}^m & h_{23}^m & \cdots & \vdots \\ \vdots & & \ddots & \ddots & \\ h_{N_R 1}^m & \cdots & & h_{N_R N_T - 1}^m & h_{N_R N_T}^m \end{bmatrix} \begin{bmatrix} s_k^{m,1} \\ s_k^{m,2} \\ \vdots \\ s_k^{m,N_T} \end{bmatrix} + \begin{bmatrix} n_k^1 \\ n_k^2 \\ \vdots \\ n_k^{N_R} \end{bmatrix}, \tag{7b}$$

where all M transmit vectors $\mathbf{s}_k^m$ are stacked into a single vector $\mathbf{s}_k$ and, correspondingly, the channel matrix $\mathbf{H}$ takes on the meaning of a large compound channel matrix. Such a situation also includes coordinated base-stations sending identical information as it is proposed as the Coordinated MultiPoint (CoMP) technique in LTE [43]. In such a case, the compound matrix $\mathbf{H}$ can take on various dimensions encompassing multiple antennas, a large range of frequencies (OFDMA), multiple codes (CDMA) as well as multiple users at the same time. In essence, all these effects are of linear nature and quite natural in a multi-user scenario. Equalizers are thus offering a great potential to remove undesired interference effects. In the end, all interference scenarios can be modeled employing the single canonical form

$$\mathbf{r} = \mathbf{H}\mathbf{s} + \mathbf{n}, \tag{8}$$

just with different interpretations of the channel matrix $\mathbf{H}$, thus substantially facilitating further analyses. Figure 3 illustrates such a canonical transmission model that will be the basis for further investigations. An equalizer matrix $\mathbf{F}$ aims to compensate for the interference of the channel $\mathbf{H}$, altering the noise at the same time. Often, a nonlinear device (slicer) is placed behind the equalizer to produce a so-called hard-decision onto the allowed signal alphabet of $\mathbf{s}_k$.

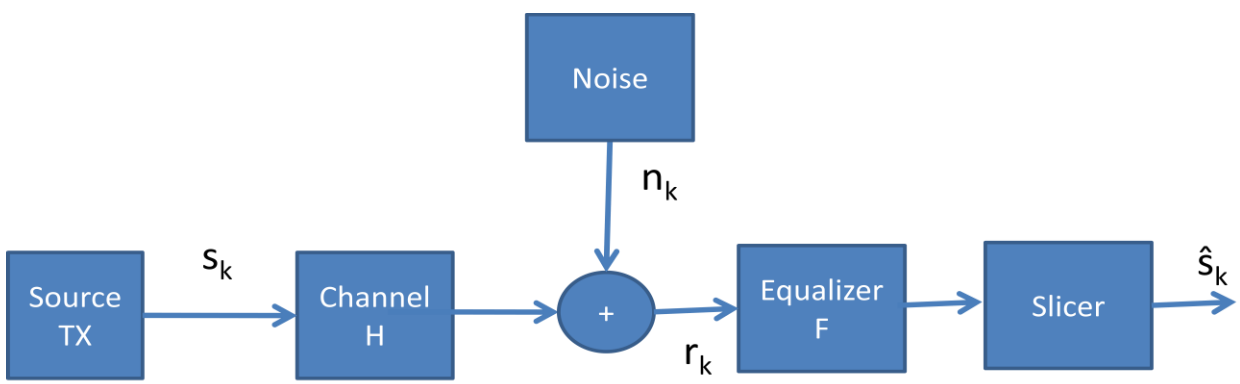

Figure 3: A canonical transmission model containing a channel $\mathbf{H}$, additive noise $\mathbf{n}_k$, equalizer $\mathbf{F}$, and a hard decision (slicer) to recover the original transmit sequence $\mathbf{s}_k$.

Some Classic Results and Basic Properties: Back to School

While apparently simple and straightforward in mathematical formulation (i.e. the equalizer can be simply viewed as the inverse of the channel matrix), equalizer solutions can become difficult when causal and acausal (non-causal or anti causal) solutions need to be distinguished.

Let us consider a classical equalizer problem. Assume for example, that the discrete-time channel transfer function $H(q^{-1})$ of a single-antenna system is given by:

$$H\left(q^{-1}\right)=\frac{20}{9}\left(q-0.8\right)q^{-2}\left(1.25-q\right)=-\frac{20}{9}+\frac{7}{3}q^{-1}-\frac{20}{9}q^{-2}, \tag{9}$$

where q^{-1} denotes a unit delay and we neglect additive noise at the receiver, so that we only suffer from classic inter-symbol interference.

From an equalizer given by its transfer function $F(q^{-1})$ we would require that $F(q^{-1})H(q^{-1}) = 1$, the equalizer $F(q^{-1})$ is just the inverse of the channel transfer function. This equalizer is termed as Zero Forcing (ZF) because all potential interference is forced to be zero. Such a ZF equalizer is given by

$$\begin{aligned} F\left(q^{-1}\right) &= \frac{9}{20}q^{2}(q-0.8)^{-1}(1.25-q)^{-1} \\ &= \underbrace{0.8q(q-0.8)^{-1}}_{\text{causal, stable}}+\underbrace{q(1-0.8q)^{-1}}_{\text{acausal, stable}} \end{aligned} \tag{10}$$

In Equation (10) the equalizer is split into a stable, causal and acausal part. While the inverse of the first part in Equation (10) is a simple exponentially-decaying sequence, the stable solution of the second's part inverse is of acausal nature, explicitly

$$0.8q(q-0.8)^{-1}=0.8(1-0.8q^{-1})^{-1} \quad \Rightarrow f_k^{\mathrm{c}}=\begin{cases}0.8^{k+1} & ;k\geq 0\\ 0 & ;\text{else}\end{cases} \tag{11a}$$

$$q(1-0.8q)^{-1}=(q^{-1}-0.8)^{-1} \quad \Rightarrow f_k^{\mathrm{a}}=\begin{cases}0.8^{-k-1} & ;k< 0\\ 0 & ;\text{else}\end{cases} \tag{11b}$$

Figure 4 illustrates these two parts of the equalizer solution: The causal part f_k^c is plotted in red while the acausal part f_k^a is printed in blue.

If only the causal part is inverted, the required filter is of infinite length (from $k = 0$ to ∞) and thus of high complexity once implemented as a Finite Impulse Response (FIR) filter. Cutting the impulse response at a certain point causes remaining inter-symbol interference. The acausal part can only be treated by an additional delay, that is, rather than requiring $H(q^{-1})F(q^{-1}) = 1$, we relax the condition to $H(q^{-1})F(q^{-1}) = q^{-D}$. The larger the delay D, the better the approximation, but the longer it takes to obtain a symbol decision. From this very simple but realistic example, we recognize that a feasible finite-length equalizer cannot offer a perfect interference-free solution.

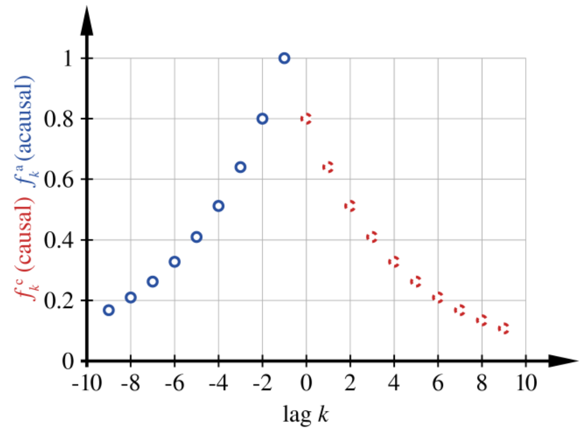

Figure 4: Causal part (red, values for $k < 0$) and acausal part (blue, values for $k \geq 0$) of the equalizer solution.

Yes, We Can: The Savior Bezout

The trick that allows for finding finite-length solutions for equalizers lies in the so-called Bezout Theorem [37-39]. Since it is mathematically evolved in its original form, we repeat it here in appropriate form for the convenience of the reader.

Bezout Theorem: Given the n transfer functions $H_1(q^{-1}),\ldots, H_n(q^{-1})$. The following equation has a finite solution, **if and only if** there is no common prime in all $H_l(q^{-1})$, $l = 1, 2, \ldots, n$.

$$\sum_{l=1}^{n} H_l(q^{-1})F_l(q^{-1}) = q^{-D}. \tag{12}$$

In other words, as long as we have at least two channels that have no zero in common, a finite-length ZF solution exists. While theoretically it is very unlikely that two independent channels share the same zero (with probability approaching zero if modeled as independent random vectors), practically, two zeros may be relatively close to each other and become hard to distinguish. Once more channels are involved the Bezout Theorem fails only if all of them share the same zero. The odds to have all channels sharing the same zero can be reduced with an increasing number of channels. Let us demonstrate the Bezout Theorem on a very simple Single-Input Multiple-Output (SIMO) transmission example. The two paths may have the channel transfer functions $H_1(q^{-1}) = h_{01} + q^{-1}h_{11}$ and $H_2(q^{-1}) = h_{02} + q^{-1}h_{12}$, the desired equalizers are called $F_1(q^{-1}) = f_{01} + q^{-1}f_{11}$ and $F_2(q^{-1}) = f_{02} + q^{-1}f_{12}$, respectively. We then find the following equation, describing our problem in vector notation:

$$\begin{aligned} \begin{bmatrix}\mathbf{H}_1 & \mathbf{H}_2\end{bmatrix} \quad \mathbf{f} &= \mathbf{d}, \\ \begin{bmatrix} h_{01} & & h_{02} & \\ h_{11} & h_{01} & h_{12} & h_{02} \\ & h_{11} & & h_{12} \end{bmatrix} \begin{bmatrix} f_{01} \\ f_{11} \\ f_{02} \\ f_{12} \end{bmatrix} &= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \end{aligned} \tag{13}$$

in which we have deliberately selected the delay $D = 1$. As we have fewer equations than unknowns, there exists in fact an infinite number of solutions (from the null space of the left-hand side matrix). Among the multitude of solutions there is a single solution, called the minimum norm solution, that exhibits the smallest Euclidian vector norm (usually favorable for implementations) and at the same time, it delivers the smallest noise impact:

$$\mathbf{f}_{\mathrm{ZF}} = \begin{bmatrix}\mathbf{H}_1 & \mathbf{H}_2\end{bmatrix}^{\mathrm{T}} \left(\begin{bmatrix}\mathbf{H}_1 & \mathbf{H}_2\end{bmatrix} \begin{bmatrix}\mathbf{H}_1 & \mathbf{H}_2\end{bmatrix}^{\mathrm{T}} \right)^{-1} \mathbf{d}. \tag{14}$$

Note that we are switching deliberately between two equivalent notations: transfer functions and matrix-vector products. As transfer functions are operators in (q^{-1}) and matrices denoted by capital bold face letters, they should be easily distinguishable. Hereafter, we present the solutions using only real-valued numbers in order to ease the comprehension of the expressions. Corresponding solutions employing complex values do also exist.

As soon as we add more elements on the channel or the equalizer, the situation remains: there are always more unknowns than equations and thus the solution of the linear set of equations exists. In summary, if the equalizer filter is at least as long as the channel impulse response, then a finite ZF solution exists. Figure 5 illustrates the setup for a SIMO system with two receive antennas. In the absence of noise, the computed signal $\hat{s}_k = s_k$.

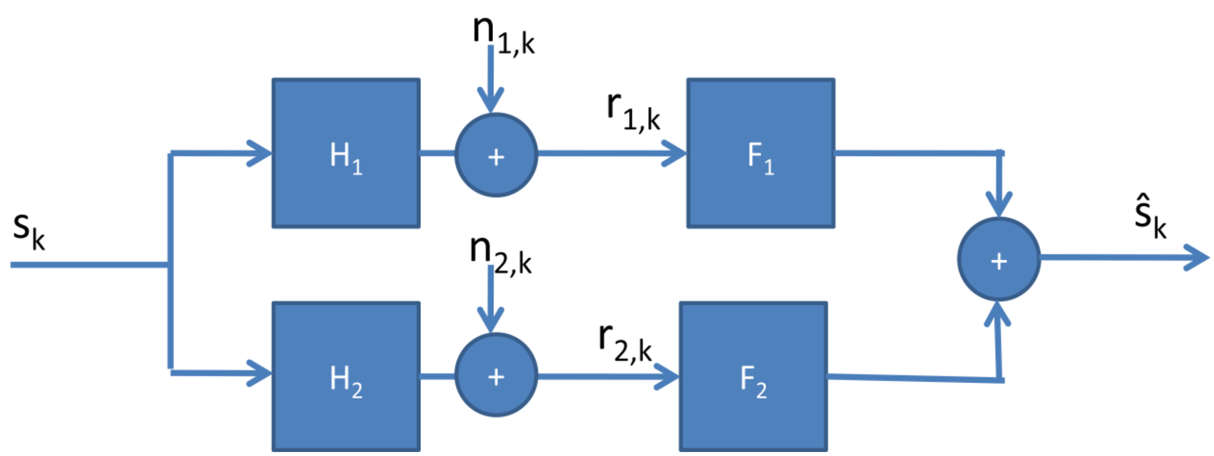

Figure 5: A Single-Input Multiple-Output (SIMO) transmission system (in this case with two receive antennas) satisfies the Bezout Theorem.

Although such a solution is quite satisfactory when at least two receive antennas are present, one may not be satisfied with it once a low-cost single-antenna receiver is employed. But even then, potential solutions do exist. One of them is called Decision Feedback Equalizer (DFE) [6-8] in which the (tentative) decision $\hat{s}_k$ is fed back into the equalizer, acting as a second path. Figure 6 presents the concept indicating the correct transmitted sequence s_k as an input. As the computation of $\hat{s}_k$ will take some time, a delay, say of L steps, is required. In order to have the estimated sequence $\hat{s}_k$ as input signal to the feedback path, a particular noise sequence is introduced. This noise at the input of the feedback path is not an observation noise but a decision noise in case a wrong decision has been made and zero otherwise. By such a technique, an artificial second path $H_2(q^{-1})=q^{-L}$ has been introduced. The drawback of DFE structures is their tendency to error propagation [47]. In case $F_2(q^{-1})$ has many taps, an error introduced once will remain active for a long time. This in turn can facilitate the generation of new errors and the DFE structure can become unstable [35]. Newer techniques introduce additional robust Viterbi decoding techniques in the feedback path to ease the error propagation problem [40]. Although we address in our example only DFE structures in the time domain, they can equally be applied in the spatial domain [29] or in multiuser scenarios. There, they are typically addressed as successive interference cancellers [48, 49].

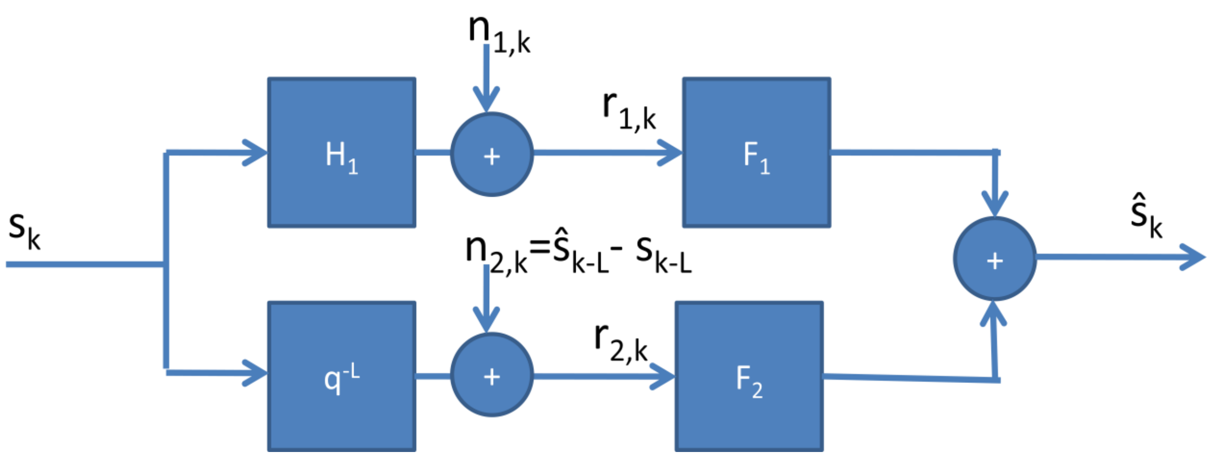

Figure 6: A Decision Feedback Equalizer (DFE) satisfies the Bezout Theorem.

An alternative to DFE structures without the problem of error propagation is the so-called fractionally-spaced oversampled equalizer (*T/2, T/4*–spaced) or polyphase equalizer [10]. Figure 7 introduces graphically the basic idea. By selecting the even (index) part of the channel $H(q^{-1})$ in the upper branch and the odd part in the lower branch two –in general co-prime– functions are generated. In the absence of noise, finite-length equalizer filters $F_1(q^{-1})$ and $F_2(q^{-1})$ can be selected to guarantee that $\hat{s}_k = s_k$. The oversampled structure, however, generates a new problem. Due to the oversampling process, the noise at the input is no longer guaranteed to be a white random process but it is correlated instead. This needs to be considered when computing the optimal solution [10].

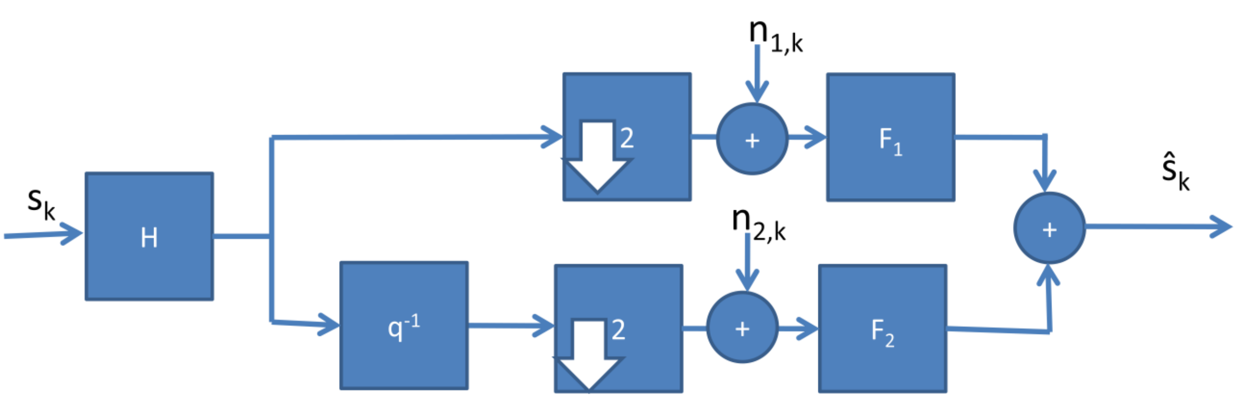

Figure 7: A *T/2*-spaced oversampled equalizer structure satisfies the Bezout Theorem.

A final application of the Bezout Theorem is given in the problem of blind channel estimation. Differently to previous schemes, the signal s_k does not even need to be partially known (training) at the receiver end, offering a better usage of the available bandwidth. Figure 8 shows an example. Consider again two available channels, for example by having two receive antennas or by fractionally sampling. Under certain requirements of the channels (Bezout condition) and persistent excitation of the transmitted signal [20, 21], it can be shown that, in the absence of noise, the only non-trivial solution exhibiting the outgoing difference $d_k = 0$ for all time instants is given by $\boldsymbol{F}_1(q^{-1}) = \alpha H_2(q^{-1})$ and $F_2(q^{-1}) = \alpha H_1(q^{-1})$as the convolution operator is commutative. In this case, the channel is known up to a multiplicative scalar α. In many cases the corresponding equalizer can be computed even without having such a scalar value.

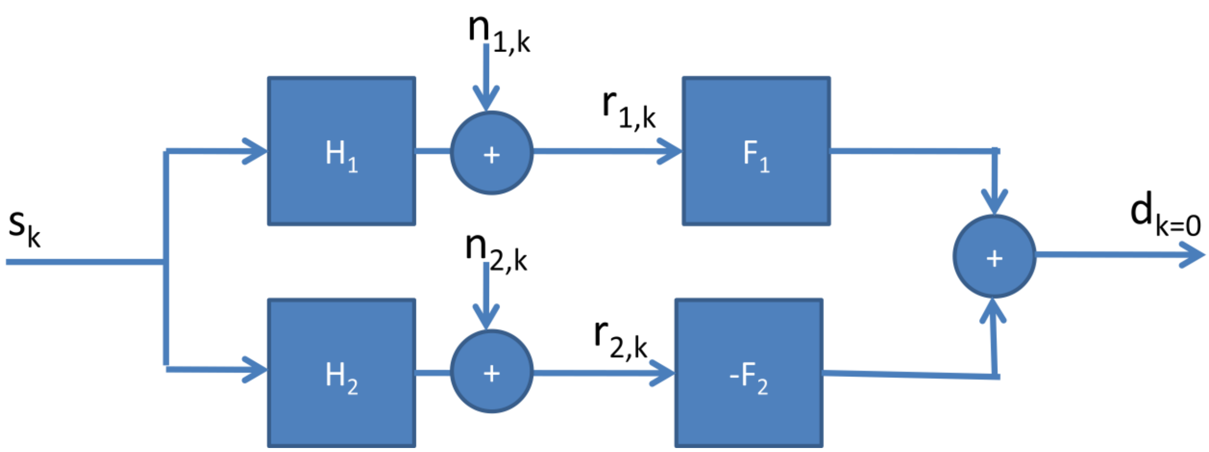

Figure 8: A blind channel estimation scheme for equalizers satisfies the Bezout Theorem.

Turning Equalizers Upside Down: Precoding

While the previous concepts seem to work well once there are more receive antennas than transmit antennas, we should also consider the opposite case, which is typical in a downlink scenario. Here we transmit from the base-station, equipped with several antennas, to the cell phone that typically can only equip a single antenna (as of time of this writing only very few cell phones with two antennas do exist!). In this case the equalizers can be located at the transmitter as plotted in Figure 9 and are called precoders. As before, a common zero of the channels transfer function forbids a finite-length solution but is expected to be the exception. Utilizing more degrees of freedom, for example by employing more antennas, can ease such a problem.

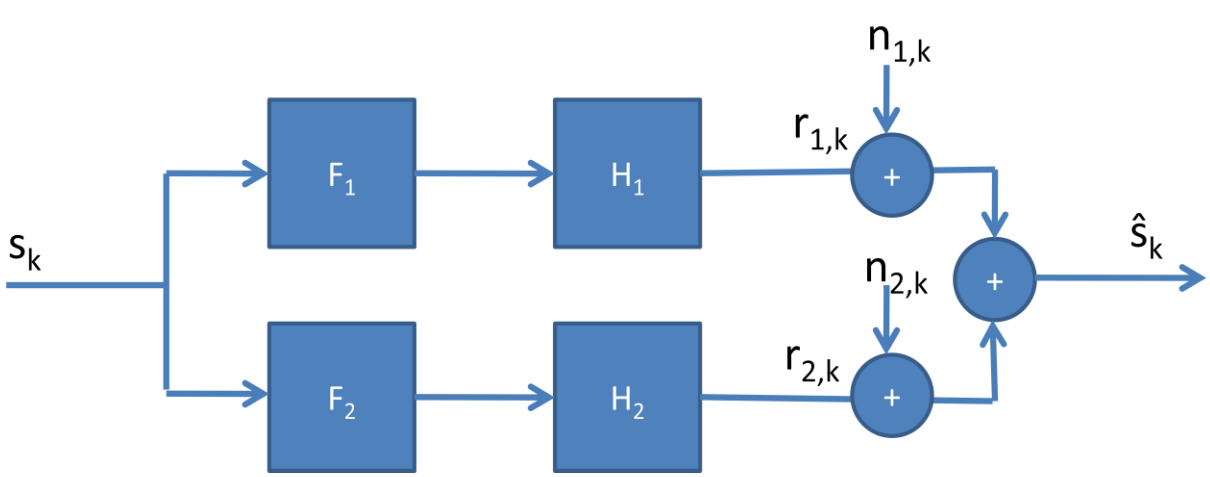

Figure 9: An equalizer structure as a precoder satisfies the Bezout Theorem.

The advantage of such a structure is obvious: the ZF solution is not corrupted by the additive noise and thus it is optimal for reception. However, the main disadvantage is that the channel now needs to be known at the transmitter. In case of the so-called Time Division Duplex (TDD) transmission mode, the same channel is being employed in both transmission directions, and due to channel reciprocity, the transmitter also knows the channel. Therefore, it is possible to apply such a precoding scheme without requiring a feedback channel. In conventional Frequency Division Duplex (FDD) transmission schemes, however, the channel is only known at the receiver and informing the transmitter could put a too-large burden on the feedback channel. Alternatively, quantized equalizer versions have been used. In this case, the transmitter selects the best performing equalizer from a limited set of equalizers known to both ends. The feedback channel is then only loaded by a few bits, indexing the selected equalizer solution from this set. In UMTS HSPA this information is called the PreCoding Indicator (PCI), in UMTS LTE it is called Precoding Matrix Indicator (PMI) and typically consists of 3-4 bits. Such a quantized version is certainly not providing an optimal solution and the receiver will apply another equalizer on top of the precoder. However, the precoder provides an important advantage in easing the problem at the receiver because the Signal-to-Noise Ratio (SNR) is considerably improved at the receiver side. We will comment more on precoding techniques in our conclusions as they are currently being seen as a potential solution for increasing capacity significantly in cellular systems.

ZF-Equalizers Improve the SIR, MMSE Equalizers Improve the SINR

After demonstrating how "easy" it is to find finite-length equalizer solutions, the reader might wonder what is the improvement offered by such equalizers. For communication systems the natural bound is the channel capacity as it was defined by Shannon. Given that this bound is directly dependent on the SNR, the challenge is typically considered to design a filter that guarantees maximum SNR. In case the finite-length equalizer satisfies the Bezout Theorem and the noise is uncorrelated, then the resulting solution maximizes the SNR. However, as the various interference scenarios previously discussed have shown, the transmission model is in general given by $\mathbf{r} = \mathbf{H}\mathbf{s} + \mathbf{n}$, where we deliberately omitted the time index k. We thus have to answer the question, what filter $\mathbf{f}$ applied on $\mathbf{r}$ decreases the interference as much as possible. The answer can be found better in terms of Signal-to-Interference and Noise Ratio (SINR). Applying the equalizer $\mathbf{f}$ on $\mathbf{r}$ causes**:**

$$\begin{aligned}\mathbf{f}^{\mathrm{T}}\mathbf{r} &= \mathbf{f}^{\mathrm{T}}\mathbf{H}\mathbf{s} + \mathbf{f}^{\mathrm{T}}\mathbf{n} \\ &= \mathbf{f}^{\mathrm{T}}\left[\mathbf{h}_1 \quad \cdots \quad \mathbf{h}_m \quad \cdots \quad \mathbf{h}_M\right]\mathbf{s} + \mathbf{f}^{\mathrm{T}}\mathbf{n} \\ &= \underbrace{\mathbf{f}^{\mathrm{T}}\mathbf{h}_m\mathbf{e}_m^{\mathrm{T}}\mathbf{s}}_{\text{signal}} + \underbrace{\mathbf{f}^{\mathrm{T}}\left[\mathbf{h}_1 \quad \cdots \quad \mathbf{h}_{m-1}\mathbf{0}_m\mathbf{h}_{m+1} \quad \cdots \quad \mathbf{h}_M\right]\mathbf{s}}_{\text{interference}} + \underbrace{\mathbf{f}^{\mathrm{T}}\mathbf{n}}_{\text{filtered noise}},\end{aligned} \tag{15}$$

where we applied a unit vector $\mathbf{e}_m^T = [0, \ldots, 0, 1, 0, \ldots, 0]$ with a single one at position m to detect the m-th component of transmit vector $s_m = \mathbf{e}_m^T\mathbf{s}$.

Assuming unit power for the data symbols, and denoting the noise variance by $N_0 = E[|\mathbf{n}_k|^2]$, we find the SINR at the receiver input, i.e., before the equalizer, as

$$\mathrm{SINR}_{\mathrm{be}}=\frac{\|\mathbf{h}_m\|_2^2}{\|\mathbf{H}\|_{\mathrm{F}}^2-\|\mathbf{h}_m\|_2^2+N_R N_o} \quad (16)$$

while afterwards it reads

$$\mathrm{SINR}_{\mathrm{ae}}=\frac{\left|\mathbf{f}^{\mathrm{T}}\mathbf{h}_m\right|^2}{\left\|\mathbf{f}^{\mathrm{T}}\mathbf{H}\right\|_2^2-\left|\mathbf{f}^{\mathrm{T}}\mathbf{h}_m\right|^2+N_o\mathbf{f}^{\mathrm{T}}\mathbf{f}}\underset{\mathrm{ZF}}{=}\frac{1}{N_o\mathbf{f}^{\mathrm{T}}\mathbf{f}} \quad (17)$$

where the last part is only true if a ZF solution can be obtained [33]. Assuming such an exact finite-length solution **f**, all interference has been cancelled and the SNR has also been improved. However, the situations in modern cellular systems may become quite challenging for a linear filter to ensure a ZF solution. Even the optimal solution may still exhibit some remaining interference. We therefore had to consider the SINR rather than the SNR as the figure of merit in Equations (16) and (17). As we cannot expect the (remaining) interference to be Gaussian as noise, maximizing capacity is no longer equivalent to maximizing SINR. But due to the absence of any better idea, maximizing SINR is usually considered as the next best (sub-)optimal solution. In order to maximize $\mathrm{SINR}_{\mathrm{ae}}$ as shown in Equation (17), a couple of mathematical tricks are needed. First of all, we recognize that the general solution (left-hand side of Equation (17)) is a ratio in the equalizer solution **f**, thus the absolute value of **f** does not matter. Maximizing

$$\mathrm{SINR}_{\mathrm{ae,max}}=\max_{\mathbf{f}}\frac{\left|\mathbf{f}^{\mathrm{T}}\mathbf{h}_m\right|^2}{\left\|\mathbf{f}^{\mathrm{T}}\mathbf{H}\right\|_2^2-\left|\mathbf{f}^{\mathrm{T}}\mathbf{h}_m\right|^2+N_o\mathbf{f}^{\mathrm{T}}\mathbf{f}} \quad (18)$$

is not so simple, but with the aid of a monotone mapping, we find

$$\frac{\mathrm{SINR}_{\mathrm{ae,max}}}{\mathrm{SINR}_{\mathrm{ae,max}}+1}=\max_{\mathbf{f}}\frac{\left|\mathbf{f}^{\mathrm{T}}\mathbf{h}_m\right|^2}{\left\|\mathbf{f}^{\mathrm{T}}\mathbf{H}\right\|_2^2+N_o\mathbf{f}^{\mathrm{T}}\mathbf{f}}. \quad (19)$$

The term in Equation (19) can be maximized by recognizing that it is a Rayleigh quotient and we find the solution immediately

$$\mathbf{f}_{\max}=\alpha\left(\mathbf{H}^{\mathrm{T}}\mathbf{H}+N_o\mathbf{I}\right)^{-1}\mathbf{h}_m. \quad (20)$$

Here α is an arbitrary factor. For α such a solution is also known as Minimum Mean Squared Error (MMSE) solution [34]. If we ignore the noise, i.e., $N_0 = 0$, then we obtain the ZF solution. In general it can be shown [41] that Equation (20) maximizes the SINR of the receiver and at the same time, it is the best equalizer in the mean square sense. Analogously, it can be shown that for $N_0 = 0$, Equation (20) maximizes the SIR and it is equivalent to the ZF solution. In [41] it is even shown that both solutions are equivalent to a specific weighted Least Squares (LS) solution.

A Typical Open Interference Problem: Two Users and Two Base-Stations

The presentation so far may make the reader believe that all problems involving equalizers have been already solved. We thus present here a simple but typical problem that is not solved yet.

Figure 10 shows a typical scenario: two users are served by one base-station each. But as they are sharing the same spectrum, the base-station signals also act as interference to the other user. Even more, as the base-stations utilize multiple antennas to transmit high data rates to their own user they also generate spatial self-interference. Let us consider that both base-stations can employ precoding filters as well as deciding the number of streams (also called spatial layer) they send towards their respective users:

$$\begin{aligned}\mathbf{r}^{(1)} &= \mathbf{H}_{11}\sum_{n=1}^{N_T}\mathbf{p}_{\mathrm{n}}^{(1)}s_{\mathrm{n}}^{(1)} + \mathbf{H}_{12}\sum_{n=1}^{N_T}\mathbf{p}_{\mathrm{n}}^{(2)}s_{\mathrm{n}}^{(2)} + \mathbf{n}^{(1)} \\ &= \mathbf{H}_{11}\mathbf{P}^{(1)}\mathbf{s}^{(1)} + \mathbf{H}_{12}\mathbf{P}^{(2)}\mathbf{s}^{(2)} + \mathbf{n}^{(1)}, \qquad (21a)\\ \mathbf{r}^{(2)} &= \mathbf{H}_{22}\sum_{n=1}^{N_T}\mathbf{p}_{\mathrm{n}}^{(2)}s_{\mathrm{n}}^{(2)} + \mathbf{H}_{21}\sum_{n=1}^{N_T}\mathbf{p}_{\mathrm{n}}^{(1)}s_{\mathrm{n}}^{(1)} + \mathbf{n}^{(2)} \\ &= \mathbf{H}_{22}\mathbf{P}^{(2)}\mathbf{s}^{(2)} + \mathbf{H}_{21}\mathbf{P}^{(1)}\mathbf{s}^{(1)} + \mathbf{n}^{(2)}. \qquad (21b)\end{aligned}$$

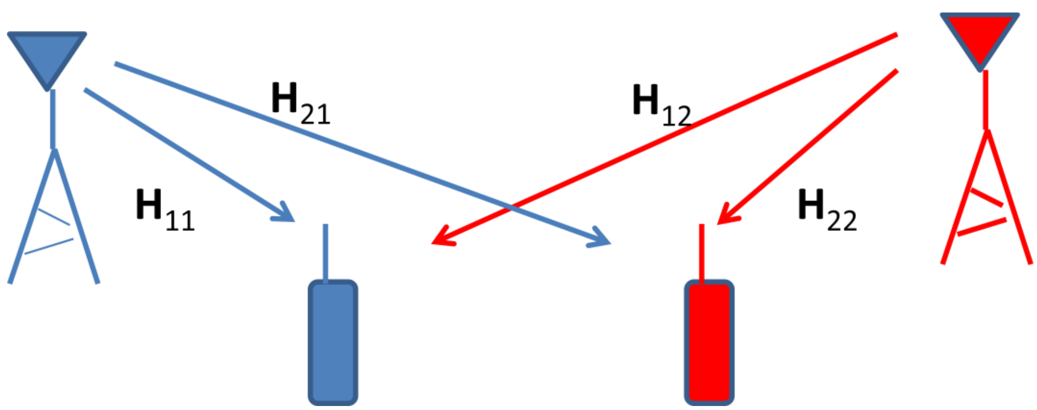

Figure 10: A typical interference problem consisting of two base-stations and two users sharing the available spectrum.

Then, the SINR for detecting the stream n at both receivers reads

$$\mathrm{SNR}_{1.n} = \frac{\left|\mathbf{f}_{\mathrm{n}}^{(1)}\mathbf{H}_{11}\mathbf{p}_{\mathrm{n}}^{(1)}\right|^2}{\sigma_n^2\left\|\mathbf{f}_{\mathrm{n}}^{(1)}\right\|_2^2 + \left\|\mathbf{f}_{\mathrm{n}}^{(1)}\mathbf{H}_{11}\sum_{m=1,\neq n}^{N_T}\mathbf{p}_{\mathrm{m}}^{(1)}\right\|_2^2 + \left\|\mathbf{f}_{\mathrm{n}}^{(1)}\mathbf{H}_{12}\sum_{m=1}^{N_T}\mathbf{p}_{\mathrm{m}}^{(2)}\right\|_2^2}, \qquad (22a)$$

$$\mathrm{SNR}_{2.n} = \frac{\left|\mathbf{f}_{\mathrm{n}}^{(2)}\mathbf{H}_{22}\mathbf{p}_{\mathrm{n}}^{(2)}\right|^2}{\sigma_n^2\left\|\mathbf{f}_{\mathrm{n}}^{(2)}\right\|_2^2 + \left\|\mathbf{f}_{\mathrm{n}}^{(2)}\mathbf{H}_{22}\sum_{m=1,\neq n}^{N_T}\mathbf{p}_{\mathrm{m}}^{(2)}\right\|_2^2 + \left\|\mathbf{f}_{\mathrm{n}}^{(2)}\mathbf{H}_{21}\sum_{m=1}^{N_T}\mathbf{p}_{\mathrm{m}}^{(1)}\right\|_2^2}. \qquad (22b)$$

This set of equations offers a cornucopia of possibilities to select the equalizers $\mathbf{f}_n^{(1)}$, $\mathbf{f}_n^{(2)}$ of the stream n as well as the corresponding precoding vectors $\mathbf{p}_m^{(1)}$, $\mathbf{p}_m^{(2)}$ at both transmission ends in order to maximize the

SINR. General analytical solutions for this problem are not known. A classical strategy consists in applying an SVD to the channels, for example $\mathbf{V}_{11}\mathbf{\Sigma}_{11}\mathbf{U}_{11}{}^{H}=\mathbf{H}_{11}$, and then use the unitary matrices $\mathbf{V}_{11}=\mathbf{P}^{(1)}$ as precoding filters. This solution ignores the interference terms corresponding to the other user.

Most surprisingly, for a problem with three transmiter-receiver pairs (three users) a solution exists. Cadambe and Jafar [36] proposed to align the interference terms into a mutual subspace of lower dimension which allows for finding an orthogonal equalizer vector to this subspace and thus to ensure that the interference terms cancel out. They showed that this can work as long as maximally $N_T/2$ streams are being used. Both solutions, however, require base-stations to have full knowledge of all subchannels which is not feasible in practice. Wireless communication standards typically allow only for selecting fixed subsets of precoding matrices in order to ensure limited information in the feedback channels. Given the set of precoding solutions, the optimal equalizers can be readily computed. In the literature, such equalizers are usually referred to as interference-aware equalizers [42, 43].

Implementation Issues and Performance Measurements

In a simulation environment, it is relatively simple to measure the quality of an equalizer. The optimal solution can be computed as a reference value and compared to the solution obtained by approximations or implementation imperfections. Once the equalizer algorithm is implemented, it is more difficult to evaluate its performance. The Error Vector Magnitude (EVM) –also called Receive Constellation Error (RCE)– is a measure used for quantifying the performance of a digital radio transmitter or receiver. A signal sent by an ideal transmitter or acquired by an ideal receiver would have all constellation points placed at their ideal locations. However, various imperfections in the implementation (such as carrier leakage, low image rejection ratio, phase noise and more) cause the actual constellation points to deviate from such ideal locations. Loosely speaking, EVM is a measure of how far the points are from their ideal locations.

An error vector is a vector in the I-Q plane between the ideal constellation point and the point received by the receiver. In other words, it is the difference between actual received symbols and ideal symbols. EVM is the average power of the error vector, normalized to the signal power. Figure 11 shows a typical I-Q plot for a 4-QAM constellation. The center points are at $(\pm1\pm j)/\sqrt{2}$.

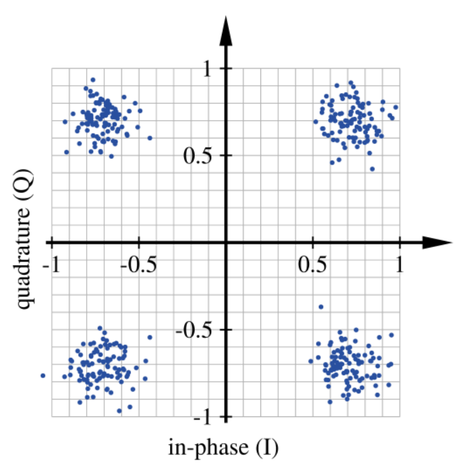

Figure 11: EVM measurement on a 4-QAM constellation.

The EVM is equal to the ratio of the power of the error vector to the power of the reference. It is defined in decibels as:

$$\text{EVM [dB]} = 10\log_{10}\left(\frac{P_{\text{error vector}}}{P_{\text{reference}}}\right), \tag{23}$$

where $P_{\text{error vector}}$ is the Root-Mean-Square (RMS) power of the error vector and $P_{\text{reference}}$ denotes the corresponding RMS power of the reference signal (i.e. 1 for the 4-QAM example in Figure 11). For single carrier modulations, $P_{\text{reference}}$ is, by convention, the power of the outermost (the highest power) point in the reference signal constellation. More recently, for multi-carrier modulations, $P_{\text{reference}}$ is defined as the reference constellation average power.

EVM can also be defined as a percentage:

$$\text{EVM [\%]} = 100\%\sqrt{\frac{P_{\text{error vector}}}{P_{\text{reference}}}}. \tag{24}$$

EVM, as conventionally defined for single carrier modulations, is a ratio of a mean power to a peak power. Given that the relationship between the peak and mean signal power depends on the constellation geometry, different constellation types (e.g. 16-QAM and 64-QAM) subject to the same mean level of interference will report different EVM values. According to [43, 44], in a modern 4G system such as LTE, we find 17.5% for 4-QAM, 12.5% for 16-QAM, and 8% for 64-QAM.

While it seems that we have a rather straightforward solution readily available even for complex transmission systems, reality looks somehow different. Our solutions so far require the inversion of a matrix, a matrix that can be large and ill-conditioned at the same time. Such a problem is not suited for low-cost 16-bit fixed-point signal processors as these are common nowadays in cellular phones. Although there exist a cornucopia of potential solutions to compute a matrix inverse, fixed-point processors can only deal with simple matrix-vector operations. Such operations can be applied iteratively to approximate the desired solution and are usually referred to as "adaptive equalizer algorithms".

Adaptive Equalizer Algorithms

If an adaptive equalizer is employed, it is actually not necessary to explicitly compute the inverse of a matrix $\mathbf{HH}^{\mathrm{T}}$ for ZF or ($\mathbf{HH}^{\mathrm{T}} + N_o\mathbf{I}$) for MMSE, but the equalizer solution alone is the final goal. To this end, two iterative algorithms can be implemented. Starting with an initial guess (which can be all zero), the algorithm iterates

$$\hat{\mathbf{f}}_{\text{ZF},n+1} = \hat{\mathbf{f}}_{\text{ZF},n} \quad + \mu\mathbf{H}(\mathbf{e}_m - \mathbf{H}^{\mathrm{T}}\hat{\mathbf{f}}_{\text{ZF},n}), \qquad n = 1,2,3... \tag{25}$$

$$\hat{\mathbf{f}}_{\text{MMSE},n+1} = \hat{\mathbf{f}}_{\text{MMSE},n} + \mu\left(\mathbf{He}_m - \left(\mathbf{HH}^{\mathrm{T}} + N_0\mathbf{I}\right)\hat{\mathbf{f}}_{\text{MMSE},n}\right), \quad n = 1,2,3..., \tag{26}$$

until a certain quality is achieved or alternatively, an amount of complexity (number of iterations) has been used. While being very robust in terms of fixed-point implementations, the only design issue is the selection of the step-size $\mu > 0$. If selected too small, the algorithm will not converge rapidly; if selected too large, the algorithm may even become unstable and diverge. Safe but conservative upper bounds of the step-size are usually readily derived; nevertheless, finding optimal step-sizes can be a tricky business.

Many details about the design of adaptive equalizer algorithms can be found in textbooks [32, 33, 34], whereas recent results in [41, 45].

As we mentioned in the beginning of this tutorial, the channel introduces two effects, the second one being a time-varying environment. In this case, the channel needs to be continuously estimated and a new equalizer solution needs to be permanently calculated. This is by far much too complex for a cost-efficient implementation. Another form of algorithm is thus better suited, that works directly on the observations and does not require explicit knowledge of the channel beforehand. The algorithm starts again with an initial estimate (which can be zero) and iterates according to

$$\hat{\mathbf{f}}_{\text{MMSE},k+1} = \hat{\mathbf{f}}_{\text{MMSE},k} + \mu \mathbf{r}_k \left(s_{k-D} - \mathbf{r}_k^{\text{T}} \hat{\mathbf{f}}_{\text{MMSE},k} \right), \quad k = 1, 2, 3, ..., \tag{27}$$

for every time instant k, that is, whenever new observations are available. While such algorithms are very suitable for practical implementations, their behavior is data-dependent, that is, they may react differently on two different data streams. An important property of such algorithms is the convergence, which is often analyzed in probabilistic terms as transmitted data appear as a random sequence. While such a convergence in the mean square sense is a very useful property, it still does not guarantee that the algorithm never becomes unstable. Algorithms with such robustness property, that is, they cannot become unstable even for the worst-case sequence (in that case they may not improve either), are of interest in safety relevant environments. A robust algorithm is, however, only known for the MMSE solution (27). For ZF only approximate solutions are known. See [45] for more details.

Challenges and Open Issues

In essence, adaptive equalizers derive the required information for their optimal operation out of their observations, which makes them being autarkic systems. However, new research has shown that substantial gain on transmitted data rates can be achieved by applying a so-called precoding filter also at the transmit base-stations. For optimal solutions, the required information for selecting optimal filters must be derived from the cellular mobiles and transferred back to the neighboring base-station, which in turn distributes the information among other base-stations to jointly find an optimal global solution. Such methods require thus a very high load on information exchange prior their data rate improvements. A compromise is to send only a few information bits about other users to limit the data flow. However, open issues are how to design such systems that they are able to gain most of the potential benefit. In particular, if the matrices that need to be inverted become too large (currently, a size greater than a hundred is a big challenge) such a small device of limited complexity may not be able to support the challenging computation of correct equalizer solutions.

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