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Permutations, Signs, and Sum Ranges †

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† Dedicated to M. J. Chasco on the occasion of her 65th birthday.

Abstract: The sum range $\mathrm{SR}[\mathbf{x}; X]$, for a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ of elements of a topological vector space X, is defined as the set of all elements $s \in X$ for which there exists a bijection (=permutation) $\pi : \mathbb{N} \to \mathbb{N}$, such that the sequence of partial sums $(\sum_{k=1}^{n} x_{\pi(k)})_{n\in\mathbb{N}}$ converges to s. The sum range problem consists of describing the structure of the sum ranges for certain classes of sequences. We present a survey of the results related to the sum range problem in finite- and infinite-dimensional cases. First, we provide the basic terminology. Next, we devote attention to the one-dimensional case, i.e., to the Riemann–Dini theorem. Then, we deal with spaces where the sum ranges are closed affine for all sequences, and we include some counterexamples. Next, we present a complete exposition of all the known results for general spaces, where the sum ranges are closed affine for sequences satisfying some additional conditions. Finally, we formulate two open questions.

Keywords: series; permutation; convergence; sum range

MSC: 54C35; 54E15

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1. Basic Definitions

We write $\mathbb{N}$ for the set $\{1, 2, \ldots\}$ of natural numbers with its usual order, and

$$\mathbb{N}_n := \{k \in \mathbb{N} : k \le n\}, \; n = 1, 2, \ldots .$$

For any set I, a bijection $\sigma : I \to I$ is called a permutation of I; we denote by $\mathbb{S}(I)$ the set of all permutations of I.

For a semigroup $(X, +)$, a natural number n, and a finite sequence $x_k \in X, k = 1, \ldots, n$

- The sum $\sum_{k=1}^{n} x_k$ is defined in the usual way;
- It is known that if $(X, +)$ is Abelian, then for every permutation $\sigma \in \mathbb{S}(\mathbb{N}_n)$, the equality

$$\sum_{k=1}^{n} x_{\sigma(k)} = \sum_{k=1}^{n} x_k$$

holds.

A (formal infinite) series corresponding to a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ of elements of an additive semigroup $(X, +)$ is the sequence of partial sums

$$\left(\sum_{k=1}^{n} x_k\right)_{n\in\mathbb{N}} . \tag{1}$$

The 'multiplicative' counterpart of the similar concept would be: a (formal) infinite product corresponding to a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ of elements of a multiplicative Abelian semigroup $(X, \cdot)$ is the sequence of partial products

$$\left(\prod_{k=1}^{n} x_k\right)_{n\in\mathbb{N}}. \tag{2}$$

A topologized semigroup is a pair (X, τ), where X is a semigroup, and τ is a topology in X.

A topological semigroup is a topologized semigroup (X, τ) for which the semigroup operation is τ-continuous.

A D-convergence space is a pair $(X, \lim)$, where X is a set, and $\lim \subset X^{\mathbb{N}} \times X$ is a relation with natural properties, see [1,2].

If $(X, \lim)$ is a D-convergence space, $\mathbf{s} = (s_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}$, and $a \in X$, then instead of $(\mathbf{s}, a) \in \lim$, we write $\lim \mathbf{s} = a$ or $\lim_n s_n = a$ and say that the sequence $\mathbf{s} = (s_n)_{n\in\mathbb{N}}$ converges to the element a.

A D-convergence semigroup is a D-convergence space $(X, \lim)$, where X is a semigroup.

A series corresponding to a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ of elements of a topologized semigroup $(X, +, \tau)$ or a D-convergence semigroup $(X, +, \lim)$ is said to be convergent in X, if there exists an element $s \in X$, such that the sequence

$$\left(\sum_{k=1}^{n} x_k\right)_{n\in\mathbb{N}}$$

converges in (X, τ), respectively, in $(X, \lim)$ to s.

If the series corresponding to a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ of elements of a topologized or D-convergence semigroup X converges to an element $s \in X$, then the element s is called a sum of the series, and we write

$$s = \sum_{k=1}^{\infty} x_k \qquad \text{or} \qquad \sum_{k=1}^{\infty} x_k = s\,.$$

Note that Bourbaki uses the notation $\mathrm{S}_{k=1}^{\infty} x_k$ instead of $\sum_{k=1}^{\infty} x_k$.

In connection with these notions, the following questions can be posed.

Question 1. *Let X be a Hausdorff topological Abelian group and $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ be a sequence of elements of X. If the series corresponding to a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ is convergent in X, and $\sigma \in \mathbb{S}(\mathbb{N})$ is a permutation, is the series corresponding to the sequence $\mathbf{x}_\sigma = (x_{\sigma(n)})_{n\in\mathbb{N}}$ convergent in X?*

Question 2. *Let X be a Hausdorff topological Abelian group and $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ be a sequence of elements of X. If the series corresponding to a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ is convergent in X, and $\sigma \in \mathbb{S}(\mathbb{N})$ is a permutation, such that the series corresponding to a sequence $\mathbf{x}_\sigma = (x_{\sigma(n)})_{n\in\mathbb{N}}$ is convergent in X too, is the equality*

$$\sum_{k=1}^{\infty} x_{\sigma(k)} = \sum_{k=1}^{\infty} x_k$$

true?

It seems that Augustin-Luis Cauchy (1789–1857) was the first who noticed (in 1833) that the answer to Question 1, in the case of the set $(\mathbb{R}, +)$ of real numbers with the usual notion of convergence, is negative.

Namely, Cauchy (pp. 57–58, [3]), first indicated (without giving any reference) a proof of the assertion that the series corresponding to the sequence $x_n = (-1)^{n+1}\frac{1}{n}$,

$n = 1, 2, \ldots$ converges in $\mathbb{R}$ and then describes a permutation $\sigma : \mathbb{N} \to \mathbb{N}$, such that the series corresponding to the sequence $x_{\sigma(n)},\ n = 1, 2, \ldots$ does not converge in $\mathbb{R}$.

The second was Peter Lejeune-Dirichlet (1805–1859), who noticed in his 1837 paper (p. 3, [4]) (without any reference either) that the answers to both Questions 1 and 2 were negative. See Remark 1 below about Dirichlet's statements.

Motivated by the abovementioned negative answers to Questions 1 and 2, for any sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ of elements of a topologized semigroup or a D-convergence semigroup $(X, +)$, we define the subsets

$$\mathfrak{P}[\mathbf{x}; X], \qquad \mathfrak{C}[\mathbf{x}; X]$$

of $\mathbb{S}(\mathbb{N})$ and the subsets

$$\mathrm{SR}[\mathbf{x}; X], \qquad \mathrm{LPR}[\mathbf{x}; X]$$

of X as follows:

- A permutation $\pi : \mathbb{N} \to \mathbb{N}$ is in $\mathfrak{P}[\mathbf{x}; X]$, if and only if the series corresponding to $(x_{\pi(n)})_{n\in\mathbb{N}}$ is convergent in X.
- A permutation $\pi : \mathbb{N} \to \mathbb{N}$ is in $\mathfrak{C}[\mathbf{x}; X]$, if and only if some subsequence of the sequence $(\sum_{k=1}^{n} x_{\pi(k)})_{n\in\mathbb{N}}$ converges in X.
- An element $t \in X$ is in $\mathrm{SR}[\mathbf{x}; X]$, if and only if $\exists \pi \in \mathfrak{P}[\mathbf{x}; X]$, such that $t = \sum_{k=1}^{\infty} x_{\pi(k)}$.
- An element $t \in X$ belongs to $\mathrm{LPR}[\mathbf{x}; X]$, if and only if $\exists \pi \in \mathfrak{C}[\mathbf{x}; X]$ such that some subsequence of the sequence $(\sum_{k=1}^{n} x_{\pi(k)})_{n\in\mathbb{N}}$ converges in X to t.

The set $\mathrm{SR}[\mathbf{x}; X]$ is called the sum range for the sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ (see Definition 2.1.1, [5]), and the set $\mathrm{LPR}[\mathbf{x}; X]$ is called the limit-point range of the series corresponding to the sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ (see Definition 3.2.1, [5], where this set is denoted by $\mathrm{LPR}(\sum_{k=1}^{\infty} x_k)$).

In (p. 95, [6]), instead of $\mathrm{LPR}[(x_n)_{n\in\mathbb{N}}; X]$, the notation $\mathfrak{C}(\sum_n x_n; X)$ is used.

Evidently,

$$\mathrm{SR}[\mathbf{x}; X] \subset \mathrm{LPR}[\mathbf{x}; X]. \tag{3}$$

It may be that for a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$, the set $\mathfrak{P}[\mathbf{x}; X]$ (respectively, the set $\mathfrak{C}[\mathbf{x}; X]$) is empty, in which case, $\mathrm{SR}[\mathbf{x}; X] = \emptyset$ (respectively $\mathrm{LPR}[\mathbf{x}; X] = \emptyset$) as well.

In the multiplicative case, of course, we need to say that a permutation $\pi : \mathbb{N} \to \mathbb{N}$ belongs to $\mathfrak{P}(\mathbf{x})$, if and only if the infinite product corresponding to $(x_{\pi(n)})_{n\in\mathbb{N}}$ is convergent in X, and we define the the product range

$$\mathrm{PR}[\mathbf{x}; X]$$

in a similar way.

The **sum range problem** can be stated as follows: *to describe the structure of the set* $\mathrm{SR}[\mathbf{x}; X]$ *for a sequence* $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ *of elements of a topologized semigroup* $(X, +, \tau)$ *or of a D-convergence semigroup* $(X, +, \lim)$.

Similarly, we can state the **product range problem** as follows: *to describe the structure of the set* $\mathrm{PR}[\mathbf{x}; X]$ *for a sequence* $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ *of elements of a topologized semigroup* $(X, \cdot, \tau)$ *or of a D-convergence semigroup* $(X, \cdot, \lim)$.

Let us first comment on the case of the set of extended real numbers $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ with the usual order, addition, and notion of convergence.

For a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$, for which the set $\{n \in \mathbb{N} : x_n < 0\}$ is finite, the series corresponding to a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ is always convergent in $\overline{\mathbb{R}}$; so, the expression

$$\sum_{k=1}^{\infty} x_k$$

is always defined.

Surely the following observation was known much earlier, but it is precisely formulated in one of the first papers [7] written by Maurice Fréchet (1878–1973) in 1903.

Proposition 1. *Let $X = \overline{\mathbb{R}}_+ = \{x \in \overline{\mathbb{R}} : x \geq 0\}$ with the usual order, addition, and topology. Then, X is a compact metrizable topological Abelian monoid, which has the following properties:*

(I) *For every sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ of elements of X, the series corresponding to $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ is convergent in X.*

(II) *For every sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ of elements of X and for every permutation $\sigma : \mathbb{N} \to \mathbb{N}$, the equality*

$$\sum_{k=1}^{\infty} x_{\sigma(k)} = \sum_{k=1}^{\infty} x_k$$

holds.

(III) *For every sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ of elements of X, the sum range $\mathrm{SR}[\mathbf{x}; X]$ is a singleton.*

(IV) *For every $x \in X$, there exists a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ of elements of X for which $\mathrm{SR}[\mathbf{x}; X] = \{x\}$.*

The following 'multiplicative' analogue of Proposition 1 is true as well.

Proposition 2. *Let $X = [0, 1]$ with the usual multiplication, order, and topology. Then, X is a compact metrizable topological Abelian monoid, which has the following properties:*

(I) *For every sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ of elements of X, the infinite product corresponding to $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ is convergent in X.*

(II) *For every sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ of elements of X and for every permutation $\sigma : \mathbb{N} \to \mathbb{N}$, the equality*

$$\prod_{k=1}^{\infty} x_{\sigma(k)} = \prod_{k=1}^{\infty} x_k$$

holds.

(III) *For every sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ of elements of X, the product range $\mathrm{PR}[\mathbf{x}; X]$ is a singleton.*

(IV) *For every $x \in X$, there exists a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ of elements of X for which $\mathrm{PR}[\mathbf{x}; X] = \{x\}$.*

We adopt the following definitions.

Definition 1. *The series corresponding to a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ in a topologized semigroup $(X, +, \tau)$ or a D-convergence semigroup $(X, +, \lim)$ is called unconditionally convergent (Bourbaki says commutatively convergent [8]) in $(X, +, \tau)$, if*

$$\mathfrak{P}[\mathbf{x}; X] = \mathbb{S}(\mathbb{N})\,;$$

i.e., if for every permutation $\sigma : \mathbb{N} \to \mathbb{N}$, the series corresponding to $\mathbf{x}_\sigma = (x_{\sigma(n)})_{n\in\mathbb{N}}$ is convergent in $(X, +, \tau)$ or in $(X, +, \lim)$.

Definition 2. *The infinite product corresponding to a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ in a topologized semigroup $(X, \cdot, \tau)$ or a D-convergence semigroup $(X, \cdot, \lim)$ is called unconditionally convergent, if for every permutation $\sigma : \mathbb{N} \to \mathbb{N}$, the infinite product corresponding to $\mathbf{x}_\sigma = (x_{\sigma(n)})_{n\in\mathbb{N}}$ is convergent in $(X, \cdot, \tau)$ or in $(X, \cdot, \lim)$.*

Sometimes the series corresponding to a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ is called conditionally convergent or semi-convergent, if it converges but does not converge unconditionally. We do not use these terms.

The following statement, which in a more general setting was obtained in [9], implies that the sum range problem has an easy solution in the case of unconditional convergence.

Theorem 1. *For a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ of elements of a Hausdorff topologized Abelian semigroup $(X, +, \tau)$, the following statements are true.*

(a') If the series corresponding to $\mathbf{x}$ is convergent in $(X, +, \tau)$, and $\mathrm{SR}[\mathbf{x}; X]$ is not a singleton, then there is a permutation $\lambda : \mathbb{N} \to \mathbb{N}$, such that the series corresponding to $\mathbf{x}_\lambda = (x_{\lambda(n)})_{n\in\mathbb{N}}$ is not convergent in $(X, +, \tau)$.

(a) (Commutativity theorem) *If the series corresponding to* $\mathbf{x}$ *is unconditionally convergent in* $(X, +, \tau)$, *then* $\mathrm{SR}[\mathbf{x}; X]$ *is a singleton.*

In the next section, we consider the problem in the case of $\mathbb{R}$. We see in particular that the converse to Theorem 1(a) is true for $X = \mathbb{R}$, but it fails in general, see Remark 7.

A topological group X is called protodiscrete, if every neighborhood of the neutral element of X contains an open subgroup of X.

The following assertion shows that for protodiscrete groups, the sum range problem has an easy solution too.

Proposition 3. *Let* $(X, +, \tau)$ *be a topological group and* $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ *be a sequence of elements of* X. *Consider the statements:*

(i) *The set* $\mathrm{SR}[\mathbf{x}; X]$ *is not empty.*
(ii) *The sequence* $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ *converges in* X *to the neutral element.*
(iii) *The series corresponding to* $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ *is unconditionally convergent in* X.

Then,

(I) $(i) \Longrightarrow (ii)$.
(II) $(ii) \Longrightarrow (iii)$, *provided* $(X, +, \tau)$ *is protodiscrete, sequentially complete, and Abelian.*
(III) (See (Ch.III, Section 5, Exercise 2) [8], $(i) \Longrightarrow (iii)$ *provided* $(X, +, \tau)$ *is protodiscrete sequentially complete and Abelian.*
(IV) *If* $(X, +, \tau)$ *is protodiscrete, sequentially complete, Hausdorff, and Abelian, then* $\mathrm{SR}[\mathbf{x}; X]$ *either is empty or is a singleton.*

Proof.

(I) This is well-known and is easy to verify.

(II) We fix a permutation $\sigma : \mathbb{N} \to \mathbb{N}$ and set $u_n = x_{\sigma(n)}$, $s_n = \sum_{k=1}^{n} u_k$, $n = 1, 2, \ldots$. Since (ii) is satisfied, it is easy to verify that the sequence $\mathbf{u} = (u_n)_{n\in\mathbb{N}}$ also converges in X to the neutral element. Let us deduce from this that (s_n) is a Cauchy sequence in X. Indeed, let V be an arbitrary neighborhood of zero in X. Since X is protodiscrete, there is an open subgroup H of X with $H \subset V$. Since $\lim_n u_n = 0$, there exists $N_H \in \mathbb{N}$, such that $u_n \in H$ for each $n > N_H$. We now fix arbitrarily natural numbers n and m, such that $N_H < m < n$; then, $s_n - s_m = \sum_{k=m+1}^{n} u_k \in H \subset V$, and so, (s_n) is a Cauchy sequence in X.
Since X is sequentially complete, the sequence (s_n) converges in X, i.e., the series corresponding to $\mathbf{u} = (u_n)_{n\in\mathbb{N}}$ converges in X. Since $\sigma : \mathbb{N} \to \mathbb{N}$ was an arbitrary permutation, (II) is proved.

(III) Since (i) is satisfied, by (I), for $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$, condition (ii) is satisfied too. Hence, by (II), we obtain that (iii) is true.

(IV) Suppose that the set $\mathrm{SR}[\mathbf{x}; X]$ is not empty. Then, by (I), condition (ii) is satisfied, and then by (II), the series corresponding to $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ is unconditionally convergent in X. From this, according to Theorem 1(a), we can conclude that $\mathrm{SR}[\mathbf{x}; X]$ is a singleton.

□

To formulate a general result related to the sum ranges, let us fix one more notation that does not directly involve permutations, see (p. 95, [6]).

For sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ of elements of a topologized Abelian semigroup $(X, +, \tau)$ and for each $m = 1, 2, \ldots$, let

$$\mathrm{A_m}[\mathbf{x}; X]$$

be the closure in (X, τ) of the set

$$\{s \in X : \exists I \subset \{m, m+1, \ldots\},\ I \text{ is finite}, \quad I \neq \emptyset, \quad s = \sum_{i\in I} x_i\},$$

and

$$\mathbf{A}[\mathbf{x}; X] = \bigcap_{m=1}^{\infty} \mathrm{A_m}[\mathbf{x}; X]\,.$$

Proposition 4. (See (pp. 95–96, [6]); see also [10]) *Let X be a metrizable topological Abelian group and $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ be a sequence of elements of X for which the set $\mathrm{SR}[\mathbf{x}; X]$ is not empty. Then,*

$$\mathbf{A}[\mathbf{x}; X]$$

is a closed subgroup of X.

Moreover,

$$\mathbf{A}[\mathbf{x}; X] + s = \mathrm{LPR}[\mathbf{x}; X],$$

for every $s \in \mathrm{SR}[\mathbf{x}; X]$.

It can be said that this proposition is the only result related to the sum range, which is valid for all metrizable topological Abelian groups. In the next section, we consider the classical case of real numbers.

2. Riemann–Dini Theorem

Let us reproduce a piece from (p. 3, [4]):

> "…we respect the essential difference which exists between two kinds of infinite series. If we regard each value instead of each term or, it being imaginary, its module, then two cases can happen. Either it is possible to give a finite value which is greater than the sum of any of however many of these values or moduli, or this condition cannot be satisfied by any finite number. In the first case, the series always converges and has a completely defined sum regardless how the series terms are ordered, …"

It follows that the following result was discovered by Dirichlet in 1837.

Theorem 2. *Let $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ be a sequence of real or complex numbers, such that for some finite number L, we have $\sum_{k=1}^{n} |x_k| \le L,\ n = 1, 2, \ldots$; i.e., in modern terms, the series corresponding to $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ is absolutely convergent.*

Then, the series corresponding to $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ is unconditionally convergent, and $\mathrm{SR}[\mathbf{x}; \mathbb{R}]$ is a singleton.

Dirichlet continues as follows:

> "…In the second case the series can converge too but convergence is essentially dependent on the kind of order of terms. Does convergence hold for a specific order then it can stop when this order is changed, or, if this does not happen, then the sum of the series might become completely different.
>
> So, for example, of the two series made from the same terms:
>
> $$1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} - \frac{1}{\sqrt{6}} + \ldots,$$
>
> $$1 + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{7}} - \frac{1}{\sqrt{4}} + \ldots,$$
>
> only the first converges while of the following:
>
> $$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots,$$
>
> $$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \ldots,$$
>
> both converge, but with different sums."

Remark 1. *Let us formulate Dirichlet's statements in terms of the present article. We introduce the sequences of real numbers* $\mathbf{a} = (a_n)_{n\in\mathbb{N}}$ *and* $\mathbf{c} = (c_n)_{n\in\mathbb{N}}$ *defined for a fixed* $n \in \mathbb{N}$ *by the equalities:*

$$a_n = (-1)^{n+1}\frac{1}{\sqrt{n}}, \quad c_n = (-1)^{n+1}\frac{1}{n}\ .$$

Let $\sigma : \mathbb{N} \to \mathbb{N}$ *be a mapping defined for a fixed* $n \in \mathbb{N}$ *by the equalities:*

$$\sigma(3n-2) = 4n-3,\ \sigma(3n-1) = 4n-1,\ \sigma(3n) = 2n\,.$$

Clearly, σ *is a bijection, i.e.,* $\sigma \in \mathbb{S}(\mathbb{N})$*.*

We have:

(D1) The series corresponding to the sequence $\mathbf{a} = (a_n)_{n\in\mathbb{N}}$ *converges in* $\mathbb{R}$*.*
The convergence follows from Leibniz's alternating series theorem; we have, moreover, that

$$0 < a_1 + a_2 < \sum_{k=1}^{\infty} a_k < a_1 = 1,\ \ 0 < \sum_{k=1}^{2n} a_k < \sum_{k=1}^{\infty} a_k < \sum_{k=1}^{2n-1} a_k < 1,\ \ n = 2, 3, \ldots .$$

The exact value of $\sum_{n=1}^{\infty} a_n$ *seems to be unknown.*

(D2) The series corresponding to the sequence $(a_{\sigma(n)})_{n\in\mathbb{N}}$ *does not converge in* $\mathbb{R}$*.*
This needs little work; it can be shown that, in fact,

$$\lim_n \sum_{k=1}^{n} a_{\sigma(k)} = +\infty\,.$$

(D3) The series corresponding to the sequence $\mathbf{c} = (c_n)_{n\in\mathbb{N}}$ *converges in* $\mathbb{R}$*.*
This follows again from the alternating series theorem. The value of $\sum_{n=1}^{\infty} c_n$ *is known; it is* $\ln 2$*.*

(D4) The series corresponding to the sequence $(c_{\sigma(n)})_{n\in\mathbb{N}}$ *converges in* $\mathbb{R}$ *too, and*

$$\sum_{n=1}^{\infty} c_{\sigma(n)} = \frac{3}{2}\ln 2\,.$$

This needs more work.

As we see, Dirichlet's conclusions are correct. Now, we know that Dirichlet could consider only one sequence, either $\mathbf{a} = (a_n)_{n\in\mathbb{N}}$ *or* $\mathbf{c} = (c_n)_{n\in\mathbb{N}}$*, to obtain the same conclusions, because the following statements are true as well:*

(D2') By Riemann's Theorem 3, there exists $\sigma' \in \mathbb{S}(\mathbb{N})$*, such that the series corresponding to* $(a_{\sigma'(n)})_{n\in\mathbb{N}}$ *converges in* $\mathbb{R}$*, but*

$$\sum_{n=1}^{\infty} a_{\sigma'(n)} \neq \sum_{n=1}^{\infty} a_n\,.$$

(D4') By Dini's Theorem 4(c), there exists $\sigma'' \in \mathbb{S}(\mathbb{N})$*, such that the series corresponding to* $(c_{\sigma''(n)})_{n\in\mathbb{N}}$ *does not converge in* $\mathbb{R}$*.*

It is not clear in advance that an unconditionally convergent series of real numbers is absolutely convergent as well. We shall see (Proposition 6 below) that this is in fact true due to the following Riemann rearrangement theorem, which was first published in 1867:

Theorem 3. *Let* $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ *be a sequence of real numbers, such that the series corresponding to it is convergent, but it is not absolutely convergent. Then,* $\mathrm{SR}[\mathbf{x}; \mathbb{R}] = \mathbb{R}$*.*

Let us reproduce Riemann's (1826–1866) text:

"... Dirichlet found a way to solve this problem noting that infinite series form two essentially distinct classes: those which remain convergent if all their terms are made positive and those where this is not the case. In the first case the terms of a series can be permutated arbitrarily, while in the second case the sum of a series depends on the order of terms. In fact, let for a series from the second class the positive terms be

$$a_1, a_2, a_3, \ldots,$$

and the negatives be

$$-b_1, -b_2, -b_3, \ldots$$

Then it is clear that both of the sums $\sum a$ and $\sum b$ must be divergent; in fact, if both of them are convergent, then the given series would be convergent after making all signs of its terms the same; if only one of them is convergent, then the given series would be divergent. It is not hard to see that after appropriate permutation of terms the series may take an arbitrary given value C. In fact, let us take alternately first positive terms of the series until their sum does not exceed C, and then the negative terms until the sum will not be less than C; in this way the deviation of the sum from C will never be greater than the absolute value of the preceding term whose sign has been changed. But as the values a and b when the indices increase became infinitely small, we get that the deviation from C after sufficient continuation of the series will become arbitrarily small, and hence the series converges to the value C." (Translated from (Section 3, p. 232, [11]))

"…infinite series fall into two distinct classes, depending on whether or not they remain convergent when all the terms are made positive. In the first class the terms can be arbitrarily rearranged; in the second, on the other hand, the value is dependent on the ordering of the terms. Indeed, if we denote the positive terms of a series in the second class by

$$a_1, a_2, a_3, \ldots,$$

and the negative terms by

$$-b_1, -b_2, -b_3, \ldots$$

then it is clear that $\sum a$ as well as $\sum b$ must be infinite. For if they were both finite, the series would still be convergent after making all the signs the same. If only one were infinite, then the series would diverge. Clearly now an arbitrarily given value C can be obtained by a suitable reordering of the terms. We take alternately the positive terms of the series until the sum is greater than C, and then the negative terms until the sum is less than C. The deviation from C never amounts to more than the size of the term at the last place the signs were switched. Now, since the numbers a as well as the numbers b become infinitely small with increasing index, so do also the deviations from C. If we proceed sufficiently far in the series, the deviation becomes arbitrarily small, that is, the series converges to C." (See (pp. 226–227, [12]))

According to (p. 19, [13]) Theorem 3 "made its first appearance in the work of B. Riemann (1854). It was not until after Riemann's death that a small gap in his reasoning was discovered and closed by U. Dini (1868)." Here, B. Riemann (1854) is [14], and U. Dini (1868) is [15]. We could not find any mention of 'a small gap' either in [15] or in [16].

These two theorems amount to a complete solution of the sum range problem for $\mathbb{R}$.

Proposition 5. *Let* $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ *be a sequence of real numbers. Then,*

(I) *One of the following must be true:*
- (a) $\mathrm{SR}[\mathbf{x};\mathbb{R}] = \varnothing$.
- (b) $\mathrm{SR}[\mathbf{x};\mathbb{R}]$ *is a singleton.*
- (c) $\mathrm{SR}[\mathbf{x};\mathbb{R}] = \mathbb{R}$.

(II) *Case (b) takes place if and only if the series corresponding to $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ is unconditionally convergent.*

Proof.

(I) Suppose that $\mathrm{SR}[\mathbf{x};\mathbb{R}] \neq \emptyset$. We fix a permutation $\pi : \mathbb{N} \to \mathbb{N}$, such that the series corresponding to $(x_{\pi(n)})_{n\in\mathbb{N}}$ is convergent. We write $y_n = x_{\pi(n)}, n = 1, 2, \ldots$ and $\mathbf{y} = (y_n)_{n\in\mathbb{N}}$. It is clear that

$$\mathrm{SR}[\mathbf{x};\mathbb{R}] = \mathrm{SR}[\mathbf{y};\mathbb{R}]. \tag{4}$$

If the series corresponding to $\mathbf{y} = (y_n)_{n\in\mathbb{N}}$ is absolutely convergent, then by Theorem 2, we have that $\mathrm{SR}[\mathbf{y};\mathbb{R}]$ is a singleton, and by equality (4), we have that the set $\mathrm{SR}[\mathbf{x};\mathbb{R}]$ is a singleton too.

If the series corresponding to $\mathbf{y} = (y_n)_{n\in\mathbb{N}}$ is not absolutely convergent, then by Theorem 3, we have $\mathrm{SR}[\mathbf{y};\mathbb{R}] = \mathbb{R}$, and by equality (4), we have that the equality $\mathrm{SR}[\mathbf{x};\mathbb{R}] = \mathbb{R}$ holds, too.

(II) It remains to prove that if the set $\mathrm{SR}[\mathbf{x};\mathbb{R}]$ is a singleton, then the series corresponding to $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ is unconditionally convergent. Since $\mathrm{SR}[\mathbf{x};\mathbb{R}] \neq \emptyset$, we can fix again a permutation $\pi : \mathbb{N} \to \mathbb{N}$, such that the series corresponding to $(x_{\pi(n)})_{n\in\mathbb{N}}$ is convergent; we write $y_n = x_{\pi(n)}, n = 1, 2, \ldots$, and $\mathbf{y} = (y_n)_{n\in\mathbb{N}}$. By equality (4), we have that the set $\mathrm{SR}[\mathbf{y};\mathbb{R}]$ is a singleton too; in particular, $\mathrm{SR}[\mathbf{y};\mathbb{R}] \neq \mathbb{R}$. From this, by Theorem 3, the series corresponding to $\mathbf{y} = (y_n)_{n\in\mathbb{N}}$ is absolutely convergent. Hence, by Theorem 2, the series corresponding to $\mathbf{y} = (y_n)_{n\in\mathbb{N}}$ is unconditionally convergent; so, the series corresponding to $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ is unconditionally convergent too.

□

Proposition 6 (Riemann–Dirichlet theorem). *For a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ of real numbers, the following statements are equivalent:*

(i) *The series corresponding to $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ is unconditionally convergent in $\mathbb{R}$.*

(ii) *The series corresponding to $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ is absolutely convergent in $\mathbb{R}$.*

Proof. $(i) \Longrightarrow (ii)$. By Theorem 1(a), condition (i) implies that $\mathrm{SR}[\mathbf{x};\mathbb{R}]$ is a singleton. If (i) is satisfied, but (ii) is not true, then by Theorem 3, we should have that $\mathrm{SR}[\mathbf{x};\mathbb{R}] = \mathbb{R}$, a contradiction.

$(ii) \Longrightarrow (i)$ by Theorem 2. □

In what follows, for $x \in \mathbb{R}$, we write:

$$x^+ = \max(x, 0), \quad x^- = \max(-x, 0).$$

The following version of Theorem 3 was proved by Dini in [15] in 1868 and was included in [16] too.

Theorem 4 (Dini). *Let $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers.*

(a) *(Dirichlet) If $\sum_{n=1}^{\infty} x_n^+ < +\infty$, and $\sum_{n=1}^{\infty} x_n^- < +\infty$, then $\mathfrak{P}[\mathbf{x};\mathbb{R}] = \mathbb{S}(\mathbb{N})$;*

(b) *If $\sum_{n=1}^{\infty} x_n^+ < +\infty$, but $\sum_{n=1}^{\infty} x_n^- = +\infty$, or $\sum_{n=1}^{\infty} x_n^+ = +\infty$, but $\sum_{n=1}^{\infty} x_n^- < +\infty$, then $\mathfrak{P}[\mathbf{x};\mathbb{R}] = \emptyset$;*

(c) *If $x_n \to 0$, and $\sum_{n=1}^{\infty} x_n^+ = \sum_{n=1}^{\infty} x_n^- = +\infty$, then $\mathrm{SR}\left[\mathbf{x};\overline{\mathbb{R}}\right] = \overline{\mathbb{R}}$; moreover, there exists a permutation $\pi : \mathbb{N} \to \mathbb{N}$, such that*

$$-\infty \leq \liminf_n \sum_{k=1}^{n} x_{\pi(k)} < \limsup_n \sum_{k=1}^{n} x_{\pi(k)} \leq +\infty,$$

where the lower and the upper limits are taken in $\overline{\mathbb{R}}$.

Remark 2. *Note that:*

(1) *In Theorem 4(c), unlike in Theorem 2, it is not required in advance that the initial series be convergent. Theorem 4(c) easily implies Theorem 2, although this is not noted in [15], where, as we have noted already, the name of Riemann is not mentioned at all.*

(2) *The conclusion of Theorem 4(a) in [15] reads as follows: the series corresponding to $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ is "convergent in whatever order its terms are taken".*
As we see, Dini did not write that reorderings do not affect the sum (however, prior to the formulation of his theorem, he did point this out).

(3) *The "moreover" part of Theorem 4(c) in [15] (up to the notation) is as follows: there exists a permutation $\pi : \mathbb{N} \to \mathbb{N}$ such that the series corresponding to $(x_{\pi(n)})_{n\in\mathbb{N}}$ "will also become indeterminate".*

The following statement is related to Theorem 4; the implication $(B) \implies (C)$ is taken from (Ch. IV, Section 7, Ex. 15 [8]), where the names of Riemann and Dini are not mentioned in connection with this.

Theorem 5 (Riemann–Dini–Bourbaki)**.** *Let $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. Consider the following statements.*

(A) *The series corresponding to $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ converges in $\mathbb{R}$, but $\sum_{n=1}^{\infty} |x_n| = +\infty$.*

(B) $\lim_n x_n = 0$, *and* $\sum_{n=1}^{\infty} x_n^+ = \sum_{n=1}^{\infty} x_n^- = +\infty$.

(C) *For each two elements a and b of $\overline{\mathbb{R}}$ with $a \le b$, there is a permutation σ of $\mathbb{N}$ such that:*

 (I) $\liminf_n \sum_{k=1}^{n} x_{\sigma(k)} = a$ *and,* $\limsup_n \sum_{k=1}^{n} x_{\sigma(k)} = b$ *, where* $\liminf$ *and* $\limsup$ *are taken in* $\overline{\mathbb{R}}$,
 and

 (II) *The set of cluster points of the sequence* $(\sum_{k=1}^{n} x_{\sigma(k)})_{n\in\mathbb{N}}$ *coincides with the interval* $[a, b]$.

(D) $\mathrm{SR}[\mathbf{x}; \mathbb{R}] = \mathbb{R}$.
Then, the following implications are true:

$$(A) \implies (B) \implies (C) \implies (D) \implies (B)\,.$$

Proof. The implication $(A) \implies (B)$ is well known.

A proof of $(B) \implies (C, (I))$ is in fact contained in (Theorem 3.54 (p. 76), [17]). We present a proof of the implication $(B) \implies (C, (II))$ below.

To prove the implication $(C, (I)) \implies (D)$, we fix an arbitrary $c \in \mathbb{R}$ and apply $(C, (I))$ for $a = b = c$. We obtain a permutation σ of $\mathbb{N}$, such that $\liminf_{n\to\infty} \sum_{k=1}^{n} x_{\sigma(k)} = c$ and $\limsup_{n\to\infty} \sum_{k=1}^{n} x_{\sigma(k)} = c$. Hence, $\lim_{n\to\infty} \sum_{k=1}^{n} x_{\sigma(k)} = c$. Therefore, $c \in \mathrm{SR}[\mathbf{x}; \mathbb{R}]$.

$(D) \implies (B)$. From (D), we can find and fix a permutation $\pi : \mathbb{N} \to \mathbb{N}$, such that the series corresponding to $\mathbf{x}_\pi := (x_{\pi(n)})_{n\in\mathbb{N}}$ is convergent. Clearly, we have $\mathrm{SR}[\mathbf{x}_\pi; \mathbb{R}) = \mathrm{SR}[\mathbf{x}; \mathbb{R}]$. So, we have also that $\mathrm{SR}[\mathbf{x}_\pi; \mathbb{R}] = \mathbb{R}$. From this equality and Theorem 2, we conclude that the series corresponding to $\mathbf{x}_\pi := (x_{\pi(n)})_{n\in\mathbb{N}}$ is not absolutely convergent. So, we can apply the (already proved) implication $(A) \implies (B)$ for the sequence $\mathbf{x}_\pi := (x_{\pi(n)})_{n\in\mathbb{N}}$ and obtain that $\lim_n x_{\pi(n)} = 0$, and both of the series corresponding to $(x_{\pi(n)}^+)_{n\in\mathbb{N}}$ and $(x_{\pi(n)}^-)_{n\in\mathbb{N}}$ are divergent. Hence, we have also that $\lim_n x_n = 0$, and both of the series corresponding to $(x_n^+)_{n\in\mathbb{N}}$ and $(x_n^-)_{n\in\mathbb{N}}$ are divergent as well. □

The following example shows that the implication $(D) \implies (A)$ in Theorem 5 is false in general.

Example 1. *Let*

$$a_n = \frac{1 + 2(-1)^n}{n},\ n = 1, 2, \ldots .$$

Then,

(a) *The series corresponding to $(a_n)_{n\in\mathbb{N}}$ does not converge in $\mathbb{R}$* (in fact, $\sum_{n=1}^{\infty} a_n = +\infty$).

(b) $\lim_n a_n = 0$, and $\sum_{n=1}^{\infty} a_n^+ = \sum_{n=1}^{\infty} a_n^- = +\infty$.

(c) $\mathrm{SR}[(a_n)_{n\in\mathbb{N}}; \mathbb{R}] = \mathbb{R}$.

Proof. (a) and (b) are easy to verify. (c) follows from (b) by the implication $(B) \implies (D)$ in Theorem 5. $\square$

From the following assertion, it becomes clear that the implication $(B) \implies (C,(II))$ in Theorem 5 is a consequence of the implication $(B) \implies (C,(I))$ in the same Theorem.

Theorem 6. *Let $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. Consider the statements:*

(B1) $\lim_n x_n = 0$.

(B2) The set of cluster points in $\overline{\mathbb{R}}$ of the sequence

$$(\sum_{k=1}^{n} x_k)_{n\in\mathbb{N}}$$

coincides with the interval

$$\left[\liminf_n \sum_{k=1}^{n} x_k, \limsup_n \sum_{k=1}^{n} x_k\right],$$

where the lower and the upper limits are taken in $\overline{\mathbb{R}}$.

Then, $(B1) \implies (B2)$.

We prove Theorem 6 by means of the next two propositions, which are "series free" and may be of an independent interest.

Proposition 7. *Let (s_n) be a sequence of real numbers, such that both the sets $N_+ = \{n \in \mathbb{N} : s_n \geq 0\}$ and $N_- = \{n \in \mathbb{N} : s_n < 0\}$ are infinite. Consider the following statements:*

(1) $\lim_n(s_{n+1} - s_n) = 0$.

(2) For every sequence (β_n) of strictly positive real numbers, the sequence (s_n) has a subsequence (s_{j_n}), such that

$$0 \leq s_{j_n} < \beta_n, \ \ n = 1, 2, \ldots.$$

(3) The sequence (s_n) has a subsequence (s_{j_n}) such that

$$s_{j_n} \geq 0, \ \ n = 1, 2, \ldots, \text{ and } \lim_n s_{j_n} = 0\,.$$

Then, $(1) \implies (2) \implies (3)$.

Proof. $(1) \implies (2)$.

We fix a sequence (β_n) of strictly positive real numbers. (1) implies the existence of a sequence (k_n) of natural numbers, such that

$$k \in \mathbb{N},\, k \geq k_n \implies |s_{k+1} - s_k| < \beta_n, \quad n = 1, 2, \ldots.$$

Since N_+ is an infinite set, we have that $\{i \in N_+ : i \geq k_1\} \neq \emptyset$; so, we can define

$$l_1 := \min\{i \in N_+ : i \geq k_1\}\,.$$

We have: $l_1 \geq k_1$.

Since N_- is an infinite set as well, we have that $\{i \in N_- : i > l_1\} \neq \emptyset$; so, we can define

$$m_1 := \min\{i \in N_- : i > l_1\}\,.$$

We have: $m_1 > l_1$, $k_1 \leq j_1 := m_1 - 1 \in \mathbb{N} \setminus N_- = N_+$, and

$$0 \leq s_{j_1} < s_{j_1} - s_{m_1} = |s_{j_1} - s_{m_1}| < \beta_1\,.$$

In this way, we can inductively construct a sequence (l_n) of elements of N_+ and a sequence (m_n) of elements of N_-, such that

$$k_n \leq l_n = \min\{i \in N_+ : i \geq k_n\}\,,\, m_n = \min\{i \in N_- : i > l_n\},\text{ and } n = 2,3,\ldots.$$

Then, we have:

$$m_n > l_n,\, k_n \leq j_n := m_n - 1 \in \mathbb{N} \setminus N_- = N_+,\ \ n = 1,2,\ldots,$$

and

$$0 \leq s_{j_n} < s_{j_n} - s_{m_n} = |s_{j_n} - s_{m_n}| < \beta_n,\ n = 1,2,\ldots.$$

(2) $\implies$ (3). From (2) applied for the sequence (β_n) with $\lim_n \beta_n = 0$, we obtain a subsequence (s_{j_n}) of (s_n) for which (3) is satisfied. □

Proposition 8. *Let (t_n) be a sequence of real numbers, such that*

$$\lim_n (t_{n+1} - t_n) = 0\,.$$

Then, the set of cluster points of the sequence $(t_n)_{n\in\mathbb{N}}$ in $\overline{\mathbb{R}}$ is the interval $[a,b]$, where

$$a := \liminf t_n,\ \ b := \limsup t_n.$$

(The lower and the upper limits are taken in $\overline{\mathbb{R}}$.)

Proof. The assertion is clearly true (without the assumption that $\lim_n(t_{n+1} - t_n) = 0$), if $a = b \in \overline{\mathbb{R}}$. So, we can suppose that $a < b$.

We fix $c \in]a,b[$ and put

$$s_n = t_n - c,\ \ n = 1,2,\ldots$$

We observe that

$$\liminf_n s_n = \liminf_n t_n - c = a - c < 0,\text{ and } \limsup_n s_n = \limsup_n t_n - c = b - c > 0\,. \quad (5)$$

Clearly,

(1b) $\lim_n(s_{n+1} - s_n) = 0$, and (5) implies that

(2b) The sets $N_+ = \{n \in \mathbb{N} : s_n \geq 0\}$ and $N_- = \{n \in \mathbb{N} : s_n < 0\}$ are infinite.

So, by the implication of (1) $\implies$ (3) in Proposition 7, we obtain that (s_n) has a subsequence (s_{j_n}), such that $\lim_n s_{j_n} = 0$. Hence,

$$\lim_n t_{j_n} = c,$$

and since $c \in]a,b[$ is arbitrary, Proposition 8 is proved. □

Proof of Theorem 6. Consider the sequence $t_n = \sum_{k=1}^{n} x_k,\ n = 1,2,\ldots$ Observe that since $(B1)$ is satisfied, we have $\lim_n(t_{n+1} - t_n) = 0$. An application of Proposition 8 for (t_n) gives that $(B2)$ holds for (x_n). □

Using Theorem 5, we can prove the following rearrangement theorem:

Theorem 7. *Let $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers such that* $\mathrm{LPR}[\mathbf{x};\mathbb{R}] \neq \emptyset$. *The following statements are equivalent:*

(i) $\lim_n x_n = 0\,.$

(ii) $\mathrm{SR}[\mathbf{x};\mathbb{R}] = \mathrm{LPR}[\mathbf{x};\mathbb{R}]$.

Proof. $(i) \implies (ii)$. Take $s \in \mathrm{LPR}[\mathbf{x};\mathbb{R}]$. We can find and fix a permutation $\pi : \mathbb{N} \to \mathbb{N}$ and a strictly increasing sequence $(j_n)_{n\in\mathbb{N}}$ of natural numbers, such that the sequence $\left(\sum_{k=1}^{j_n} x_{\pi(k)}\right)_{n\in\mathbb{N}}$ converges to s. If the sequence $\left(\sum_{k=1}^{n} x_{\pi(k)}\right)_{n\in\mathbb{N}}$ converges to s, then $s \in \mathrm{SR}[\mathbf{x};\mathbb{R}]$. Suppose now that the sequence $\left(\sum_{k=1}^{n} x_{\pi(k)}\right)_{n\in\mathbb{N}}$ does not converge to s. From these properties, we conclude easily that $\sum_{k=1}^{\infty} x^+_{\pi(k)} = +\infty$, and $\sum_{k=1}^{\infty} x^-_{\pi(k)} = +\infty$.

These equalities together with (i) according to implication $(B) \implies (D)$ of Theorem 5 implies that $\mathrm{SR}[\mathbf{x};\mathbb{R}] = \mathbb{R}$; in particular, $s \in \mathrm{SR}[\mathbf{x};\mathbb{R}]$.

$(ii) \implies (i)$. The equality (ii) and the condition $\mathrm{LPR}[\mathbf{x};\mathbb{R}] \neq \varnothing$ imply $\mathrm{SR}[\mathbf{x};\mathbb{R}] \neq \varnothing$. Hence, (i) holds. □

We conclude this section with the following theorem, the second item of which can be considered as a "sign analogue" of Theorem 3.

Theorem 8. *For a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ of real numbers, the following statements are true.*

(Ia) *If $\lim_n x_n = 0$, then there exists a sequence $t_n \in \{1, -1\}$, $n = 1, 2, \ldots$, such that the series corresponding to the sequence $(t_n x_n)$ converges in $\mathbb{R}$.*

(Ib) *Chapter 6, Example 6,[18]) If $\lim_n x_n = 0$, and the series corresponding to the sequence $(|x_n|)$ does not converge in $\mathbb{R}$, then for every $s \in \mathbb{R}$, there exists a sequence $t_n \in \{1, -1\}$, $n = 1, 2, \ldots$, such that the series corresponding to the sequence $(t_n x_n)$ converges in $\mathbb{R}$ and*

$$\sum_{k=1}^{\infty} t_k x_k = s\,.$$

Proof. *(Ia)* If $\sum_{n=1}^{\infty} |x_n| < +\infty$, then the assertion is true for every sequence $t_n \in \{1, -1\}$, $n = 1, 2, \ldots$.

If $\sum_{n=1}^{\infty} |x_n| = +\infty$, then the conclusion follows from (Ib).

(Ib) A proof of (Ib) can be seen in (Chapter 6, Example 6 [18]), where the requirements of (Ib) are used.

□

Remark 3. Theorem 8(Ia) remains true for a sequence of complex numbers [19].

In a footnote of the Russian translation of (Chapter 6, Example 6 [18]), it is noted that the following variant of Theorem 8(Ib) is true as well: let $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers, such that the series corresponding to the sequence (x_n) converges in $\mathbb{C}$, but $\sum_{n=1}^{\infty} |x_n| = +\infty$. Then, for every $s \in \mathbb{C}$, there exists a sequence $t_n \in \mathbb{C}$, $|t_n| = 1$, $n = 1, 2, \ldots$, such that the series corresponding to the sequence $(t_n x_n)$ converges in $\mathbb{C}$, and $\sum_{n=1}^{\infty} t_n x_n = s$.

3. Levy–Steinitz Theorem

The sum range problem for complex numbers was treated by Paul Pierre Lévy (1886–1971) in his first article [20] written in 1905 (which contains no separately formulated theorems). The problem for 'numbers' belonging to $\mathbb{R}^d$, $d = 2, 3, \ldots$ was investigated by Ernst Steinitz (1871–1928) in his cycle of articles [21–23]. As written in [23], this problem was proposed to Steinitz by Edmund Landau (1877–1938), who included his version and proof of the Riemann rearrangement theorem (without mentioning Riemann's name) in [24], as Theorem 217.

In this section, we treat the case of Hausdorff topological vector spaces over $\mathbb{R}$.

A general statement which includes Levy's and Steinitz's results was published by Banaszczyk in [25]. To formulate this statement, we first comment on the notion of a nuclear space (whose definition is not given in [25]).

We follow [26–28]. For a nonempty subset U of a monoid $(X, +)$, let $k_U : X \to [0, 1]$ be the functional defined at $x \in X$ by the equality

$$k_U(x) = \sup\{\frac{1}{n} : n \in \mathbb{N} \text{ and } nx \notin U\}$$

(we agree that $\sup(\varnothing) = 0$). Let us call the series corresponding to a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ in a topologized Abelian monoid $(X, +, \tau)$ absolutely convergent, if

$$\sum_{n=1}^{\infty} k_U(x_n) < +\infty$$

for every τ-neighborhood U of the neutral element of $(X, +, \tau)$.

It follows from [27] that the series corresponding to a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ of

- Real numbers is absolutely convergent, if and only if $\sum_{n=1}^{\infty} |x_n| < +\infty$.
- Elements of a normed space $(X, \|\cdot\|)$ are absolutely convergent, if and only if $\sum_{n=1}^{\infty} \|x_n\| < +\infty$.
- Elements of a locally convex topological vector space X are absolutely convergent, if and only if $\sum_{n=1}^{\infty} \|x_n\| < +\infty$ for every continuous semi-norm $\|\cdot\| : X \to \mathbb{R}_+$.

The notion of a nuclear locally convex space was introduced in 1953 by Grothendieck in terms of topological tensor products. We use as a definition the second item of the following consequence of Grothendieck–Pietsch's theorem (see (Ch. IV, 10.7, Corollary 2) [28] and the text following it):

Theorem 9. *For a metrizable locally convex topological vector space X over $\mathbb{R}$, the following statements are equivalent:*

(N) X is nuclear.

(GPS) For every sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ of elements of X, such that the series corresponding to $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ is unconditionally convergent in X, we have that the same series is absolutely convergent as well.

We note that the metrizability assumption in Theorem 9 is essential only for the validity of the implication $(GPS) \Longrightarrow (N)$.

It follows from the Riemann–Dirichlet theorem that $(\mathbb{R}, +)$ with the usual topology is nuclear. This implies that the spaces $(\mathbb{R}^d, +)$, $d = 2, 3\ldots$ and $(\mathbb{R}^{\mathbb{N}}, +)$ with their usual topologies are nuclear as well. It follows that any finite-dimensional normed space is nuclear. For other examples and for the general definition of nuclearity, we refer the reader to [28].

In what follows, for a topological vector space X over $\mathbb{R}$,

- We write X^* for the (topological) dual space, which consists of all continuous linear functionals $x^* : X \to \mathbb{R}$;
- The set X^* is regarded as a vector space over $\mathbb{R}$ with the usual pointwise addition and multiplication by real scalars.

A topological vector space X is called dually separated or a DS-space, if X^* separates the points of X.

The Hahn–Banach theorem implies that a Hausdorff locally convex topological vector space X over $\mathbb{R}$ is a DS-space. There are also DS-spaces, which are not locally convex.

For a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ in a topological vector space X over $\mathbb{R}$, let

$$\Gamma_{\mathbf{x}} = \{x^* \in X^* : \sum_{n=1}^{\infty} |x^*(x_n)| < +\infty\},$$

(the set $\Gamma_{\mathbf{x}}$ is called in [29] the weak summability domain of $\mathbf{x}$), and let

$${}^{\perp}\Gamma_{\mathbf{x}} := \{x \in X : x^*(x) = 0 \ \ \forall x^* \in \Gamma_{\mathbf{x}}\}$$

be the inverse polar of $\Gamma_{\mathbf{x}}$ in X.

For a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ in a topological vector space X over $\mathbb{R}$, the set $\Gamma_{\mathbf{x}}$ is a vector subspace of X^*, while the set ${}^{\perp}\Gamma_{\mathbf{x}}$ is a closed vector subspace of X.

Let us introduce also, for a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ in a topological vector space X, Steinitz's range

$$\mathrm{StR}[\mathbf{x}; X]$$

as follows:

$$\mathrm{StR}[\mathbf{x}; X] = \{s \in X : x^*(s) \in \mathrm{SR}[(x^*(x_n))_{n\in\mathbb{N}}; \mathbb{R}] \ \forall x^* \in X^* \}.$$

For a non-empty subset A of a vector space X and an element $a \in X$, we write

$$A - a = \{x - a : x \in A\},\ A + a = \{x + a : x \in A\},\ a + A = \{a + x : x \in A\}.$$

A subset A of a vector space X over $\mathbb{R}$ or over $\mathbb{C}$ is called real affine, if

$$a_1 \in A,\, a_2 \in A,\, t \in \mathbb{R} \implies ta_1 + (1-t)a_2 \in A.$$

The empty set is affine. A non-empty subset A of a vector space X over $\mathbb{R}$ is real affine, if and only if for some $a \in A$, the set $A - a$ is a vector subspace of X.

Proposition 9. *For a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ in a topological vector space X over $\mathbb{R}$, the following statements are true.*

(1) $\mathrm{SR}[\mathbf{x}; X] \subset \mathrm{StR}[\mathbf{x}; X]$.

(2) *Ref. [30], ([Proposition 2.1(b))*

$$\mathrm{StR}[\mathbf{x}; X] \neq \emptyset,\ s \in \mathrm{StR}[\mathbf{x}; X] \implies \mathrm{StR}[\mathbf{x}; X] = s + {}^{\perp}\Gamma_{\mathbf{x}}. \tag{6}$$

In particular, $\mathrm{StR}[\mathbf{x}; X]$ *is always a closed real affine subset of X.*

(3) *Ref. [31], (Proposition 1)*

$$\mathrm{SR}[\mathbf{x}; X] \neq \emptyset,\ s \in \mathrm{SR}[\mathbf{x}; X] \implies \mathrm{StR}[\mathbf{x}; X] = s + {}^{\perp}\Gamma_{\mathbf{x}}. \tag{7}$$

Proof.

(1) This is evident.

(2) Let us show first that

$$\mathrm{StR}[\mathbf{x}; X] \subset s + {}^{\perp}\Gamma_{\mathbf{x}} \tag{8}$$

We fix $a \in \mathrm{StR}[\mathbf{x}; X]$, and we show that $a \in s + {}^{\perp}\Gamma_{\mathbf{x}}$; i.e., we need to show that $a - s \in {}^{\perp}\Gamma_{\mathbf{x}}$. To see this, we fix an arbitrary $x^* \in \Gamma_{\mathbf{x}}$, and we see that $x^*(a-s) = 0$. We can find and fix permutations $\pi : \mathbb{N} \to \mathbb{N}$ and $\sigma : \mathbb{N} \to \mathbb{N}$, such that

$$\sum_{n=1}^{\infty} x^*(x_{\pi(n)}) = x^*(s) \text{ and } \sum_{n=1}^{\infty} x^*(x_{\sigma(n)}) = x^*(a). \tag{9}$$

As $x^* \in \Gamma_{\mathbf{x}}$, the series corresponding to $(|x^*(x_n)|)_{n\in\mathbb{N}}$ converges in $\mathbb{R}$. From this by Theorem 2, we obtain that $\mathrm{SR}[(x^*(x_n))_{n\in\mathbb{N}}; \mathbb{R}]$ is a singleton. This and (9) imply:

$$x^*(s) = x^*(a).$$

Hence, $x^*(a-s) = 0$, and $(b1)$ is proved.

It remains to prove that

$$\mathrm{StR}[\mathbf{x}; X] \supset s + {}^{\perp}\Gamma_{\mathbf{x}} \tag{10}$$

We fix $y \in s + {}^{\perp}\Gamma_{\mathbf{x}}$ and $x^* \in X^*$. We need to verify that

$$x^*(y) \in \mathrm{SR}[(x^*(x_n))_{n\in\mathbb{N}}; \mathbb{R}] \tag{11}$$

First, let $x^* \in \Gamma_{\mathbf{x}}$; then, (as $y \in s + {}^{\perp}\Gamma_{\mathbf{x}}$), we have $x^*(y) = x^*(s)$. As $s \in \mathrm{StR}[\mathbf{x}; X]$, for some permutation $\sigma : \mathbb{N} \to \mathbb{N}$, we have:

$$\sum_{n=1}^{\infty} x^*(x_{\sigma(n)}) = x^*(s).$$

From this, (as $x^*(y) = x^*(s)$), we obtain:

$$\sum_{n=1}^{\infty} x^*(x_{\sigma(n)}) = x^*(y).$$

So, (11) is true in this case.

Now, let $x^* \in X^* \setminus \Gamma_{\mathbf{x}}$; then, as $\mathrm{StR}[\mathbf{x}; X] \neq \emptyset$, we have that $\mathrm{SR}[(x^*(x_n))_{n\in\mathbb{N}}; \mathbb{R}] \neq \emptyset$, and the series corresponding to $(|x^*(x_n)|)_{n\in\mathbb{N}}$ is not convergent in $\mathbb{R}$. So by Riemann's theorem, we have $\mathrm{SR}[(x^*(x_n))_{n\in\mathbb{N}};\mathbb{R}] = \mathbb{R}$; hence, $x^*(y) \in \mathbb{R} = \mathrm{SR}[(x^*(x_n))_{n\in\mathbb{N}};\mathbb{R}]$. Consequently, (11) is true in this case too.

(3) This follows from (i) and (iii).

□

Now, we are ready to formulate the result.

Theorem 10 (Wojciech Banaszczyk; Theorem 1, [25])**.** *Let X be a metrizable nuclear locally convex topological vector space over $\mathbb{R}$ and $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ be a sequence in X.*

(I) If $\mathrm{SR}[\mathbf{x}; X] \neq \emptyset$, then

$$\mathrm{SR}[\mathbf{x}; X] = s + {}^{\perp}\Gamma_{\mathbf{x}}$$

for each $s \in \mathrm{SR}[\mathbf{x}; X]$.

(II) $\mathrm{SR}[\mathbf{x}; X]$ is always a closed affine subset of X.

Proof.

(I) This can be seen in [25]; see also [5], (Ch.8, Section 3 (pp. 110–117)).

(II) This follows from (I).

□

The following statement is one of the key points for the proof of Theorem 10(I).

Proposition 10 (Lemma 6, [25])**.** *Let X be a metrizable nuclear locally convex topological vector space over $\mathbb{R}$ and $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ be a sequence in X for which $\mathrm{SR}[\mathbf{x}; X] \neq \emptyset$; then,*

$$\mathbf{A}[\mathbf{x}; X] = {}^{\perp}\Gamma_{\mathbf{x}}$$

The following surprising complement to Theorem 10(I) is true as well.

Theorem 11 (Wojciech Banaszczyk; [32])**.** *For a metrizable locally convex topological vector space over $\mathbb{R}$, the following statements are equivalent:*

(i) X is nuclear.

(ii) For every sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ in X such that $\mathrm{SR}[\mathbf{x}; X] \neq \emptyset$, the equality

$$\mathrm{SR}[\mathbf{x}; X] = s + {}^{\perp}\Gamma_{\mathbf{x}}$$

holds for each $s \in \mathrm{SR}[\mathbf{x}; X]$.

In connection with this theorem, the following question seems very natural.

Question 3 (Remark 3, [32])**.** Can condition (ii) in Theorem 11 be replaced by the following condition?

(ii') For every sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ in X, the range $\mathrm{SR}[\mathbf{x}; X]$ is always a closed affine subset of X.

The question of whether the set $\mathrm{SR}[\mathbf{x}; X]$ is always convex for every sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ in a Banach space X over $\mathbb{R}$ was posed by Banach, see (Problem 106, [33]), where a negative answer was included too. It is conjectured that the corresponding example for the case $X = L_2([0,1])$ is due to Marcinkiewicz (see an interesting story in (pp. 31–32, [5])). Our presentation of this example follows (p. 173, [34]).

Example 2. (Marcinkiewicz's Example, 1936) *For a natural number m, find the nonnegative integers m' and $m'' < 2^{m'}$, such that $m = 2^{m'} + m''$, and define the function $x_m : [0,1] \to \{0,1\}$ by the equality:*

$$x_m = 1_{\left[\frac{m''}{2^{m'}}, \frac{m''+1}{2^{m'}}\right]}.$$

Now, let $\mathbf{y} = (y_n)_{n\in\mathbb{N}}$ *be the sequence defined by*

$$y_{2n-1} = x_n,\ y_{2n} = -x_n,\ n = 1, 2, \ldots.$$

This sequence has the following properties:

(a0) *The series corresponding to* $\mathbf{y} = (y_n)_{n\in\mathbb{N}}$ *converges in* $X = L_2([0,1])$ *to zero; in particular,* $0 \in \mathrm{SR}[\mathbf{y}; X]$.
(a1) $1 \in \mathrm{SR}[\mathbf{y}; X]$.
(b) *Every element of* $\mathrm{SR}[\mathbf{y}; X]$ *is an integer-valued function.*
(c) $\mathrm{SR}[\mathbf{y}; X]$ *is not a convex subset of* $X = L_2([0,1])$.

Proof. $(a0)$ follows at once from the observation that $\lim_n \|x_n\|_2 = 0$.
$(a1)$ First, we reproduce the corresponding fragment from (p. 173, [34]): since

$$y_3 + y_5 + y_2 = y_7 + y_9 + y_3 = \cdots = 0 \tag{12}$$

(it can be verified that these equalities hold almost everywhere on $[0,1]$), it follows that the series corresponding to the sequence

$$(y_1, y_3, y_5, y_2, y_7, y_9, y_3, \ldots) \tag{13}$$

converges to 1.
From this, since the sequence (13) is indeed a permutation of the sequence $\mathbf{y} = (y_n)_{n\in\mathbb{N}}$, we obtain that $1 \in \mathrm{SR}[\mathbf{y}; X]$.
(b) This is evident.
(c) From (b), we have that $\frac{1}{2} \notin \mathrm{SR}[\mathbf{y}; X]$. From this and from $(a0)$ and $(a1)$, we conclude that the set $\mathrm{SR}[\mathbf{y}; X]$ is not convex.
□

It is known that an example of the same type as Example 2 can be constructed in any infinite-dimensional Banach space X over $\mathbb{R}$ (Corollary 7.2.1 (p. 97), [5]). It follows that if the answer to Question 3 for a Banach space X over $\mathbb{R}$ is positive, then X is finite dimensional and, hence, is nuclear. In general, the answer to this question remains open.

For a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ in an infinite-dimensional separable Hilbert space X, the sum range $\mathrm{SR}[\mathbf{x}; X]$

- May not be closed [35] (see also (Example 3.1.3 (p. 31), [5]));
- It seems to be unknown whether it may be affine and non-closed (Remark 3.1.1 (p. 32) [5]);
- It is always an analytic subset of X; see [36] (hence, not any subset of a Hilbert space "can serve as the sum range of a series", see (p. 36, [5])). However, it seems to be unknown whether a sum range is always a Borel subset of X.

Remark 4. *Let* X *and* $\mathbf{y} = (y_n)_{n\in\mathbb{N}}$ *be as in Example 2. Then,*

- $\mathrm{SR}[\mathbf{y}; X] = L_2([0,1]; \mathbb{Z})$, *see (Exercise 3.1.4, [5]), where it is noted also that the inclusion* $L_2([0,1]; \mathbb{Z}) \subset \mathrm{SR}[\mathbf{y}; X]$ *was proved by Bogdan in her MSc Thesis (Zaporozhie University, Ukraine, 1992); see also (Assertion 2, [37]).*
- *Let* $X_w := (L_2([0,1]), \mathrm{weak})$. *Then,* $\mathrm{SR}[\mathbf{y}; X_w]$, *as a set, coincides with the whole space* $X = L_2([0,1]$, *see (Exercise 3.1.6, [5]).*

Remark 5. *Theorem 10*(I) *implies that that if* X *is a metrizable nuclear locally convex topological vector space over* $\mathbb{R}$, *then, for every sequence* $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ *in* X, *such that* $\mathrm{SR}[\mathbf{x}; X] \neq \emptyset$ *and* $\Gamma_{\mathbf{x}} = \{0\}$, *we have* $\mathrm{SR}[\mathbf{x}; X] = X$. *The question of validity of a similar conclusion for the case of an infinite-dimensional separable Banach space* X *was posed in (Problem 3 (p.146), [29]). A negative answer for the case* $X = L_2([0,1])$ *was obtained in [38] (see also (Proposition 3.4.4 (p. 84), [39])), where it was shown that for Marcinkiewicz's sequence* $\mathbf{y} = (y_n)_{n\in\mathbb{N}}$, *the equality* $\Gamma_{\mathbf{y}} = \{0\}$ *holds; however, by Example 2*(c), *we have:* $\mathrm{SR}[\mathbf{y}; X] \neq X = L_2([0,1])$.

Remark 6. *The question of whether the set* $\mathrm{SR}[\mathbf{x}; X)$ *is always convex for every sequence* $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ *in a separable infinite-dimensional Hilbert space* X *over* $\mathbb{R}$ *was posed (independently from Banach) and was answered negatively by Hugo Hadwiger [40]. However, in [40], it was conjectured that for each sequence* $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$*, for which* $\mathrm{SR}[\mathbf{x}; X] \neq \emptyset$*, the sum range* $\mathrm{SR}[\mathbf{x}; X]$ *must be a co-set of an additive subgroup of* X*. We reproduce an interesting piece from (p. 32, [5]):*

> *"In this section we shall give an example of a series... whose sum range consists of two points ... This example disproves, in particular, H. Hadwiger's conjecture that the sum range of any conditionally convergent series is the coset of some additive subgroup of the space under consideration. The construction given here belongs to M. I. Kadets; its justification was first obtained independently, and about the same time, by P. A. Kornilov [37] and K. Wozniakowski (see [41]). It is interesting that similar constructions were proposed at least by two other mathematicians. A. Dvoretzky told us that many years ago he had such an example, but he, too (like M. I. Kadets), was not able to find a justification. P. Enflo constructed an example with a complete proof at about the same time [37,41] were written, but he did not publish it because I. Halperin informed him about the preprint containing the example presented below."*

Note that now we know more: for each finite subset $A \subset X$ *of an infinite-dimensional separable Banach space* X *over* $\mathbb{R}$*, there exists a sequence* $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ *in* X*, such that* $\mathrm{SR}[\mathbf{x}; X] = A$ *[42].*

Theorem 10 is applicable for finite-dimensional normed spaces (because they are nuclear); however, for them, more is also true.

Theorem 12 (Ernst Steinitz)**.** *Let* X *be a finite-dimensional normed vector space over* $\mathbb{R}$ *and* $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ *be a sequence in* X*. The following statements are true.*

$(I1)$ *If* $\mathrm{SR}[(x^*(x_n))_{n\in\mathbb{N}}; \mathbb{R}] \neq \emptyset$*, for every* $x^* \in X^*$*, then* $\mathrm{SR}[\mathbf{x}; X] \neq \emptyset$*.*
$(I2)$ $\mathrm{SR}[\mathbf{x}; X] = \mathrm{StR}[\mathbf{x}; X]$*.*
$(I3)$ *If* $\mathrm{SR}[\mathbf{x}; X] \neq \emptyset$*, and* $s \in \mathrm{SR}[\mathbf{x}; X]$*, then* $\mathrm{SR}[\mathbf{x}; X] = s + {}^{\perp}\Gamma_{\mathbf{x}}$*.*
(II) $\mathrm{SR}[\mathbf{x}; X]$ *is a real affine subset of* X*.*

Comments on the Proof. $(I1)$ and $(I2)$ are nontrivial even when $\dim(X) = 2$. According to [43], $(I1)$ was proved in [23]; however, we did not find its proof in [44] or in [5]. Only in (Theorem II, [45]) and in [46] can one find some information about this implication (see also [47]).

$(I3)$ follows from $(I2)$ and Proposition 9.
Direct proofs of $(I3)$ can be seen in [45,48] and in (Chapter 2, Section 1 (pp. 13–20), [5]).

(II) follows from $(I3)$. (II) is formulated as Lemma 4 in [49] and proved there for the first time in the English literature. Proofs of (II) appeared in Russian for the first time in [50] and in [51]. □

Surely, the first attempt to understand and simplify Steinitz's exposition was carried out by the Austrian mathematician Wilhelm Gross (Groß) (1886–1918) in his 1917 article [52]. Later, many authors were interested in the proof of Theorem 12 (II). Among them was Kurt Gödel (1906–1978), one of the most outstanding logicians of the twentieth century [53] (see a nicely written account of this work in [54]). In [54], after commenting on [52], the following was written:

> *"Other authors, among them Abraham Wald (1933)(=[55]) published a proof of the theorem which was close to the proof of Gödel, and it may well be that publication of Wald's proof lessened Gödel's interest in the publication of his own manuscript."*

Let us note also that Theorem 10(II) for $X = \mathbb{R}^{\mathbb{N}}$ was derived from Theorem 12(II) earlier by Wald [56].

Theorem 12(II) coincides with (Ch. VII, Section 3, Exercise 2(ii), [57]), where a (rather complicated) hint of its proof is also given. Surely, a complete realization of Bourbaki's claim requires a separate monograph.

Let us formulate two ingredients of the proof of Theorem 12 (II), which are of independent interest. We follow the terminology of [5,45,48].

Theorem 13. (The polygonal confinement theorem.) *There exists a sequence $C_m, m = 1, 2, \ldots$ of strictly positive constants, with $C_1 = 1$ and with $C_m < m, m = 2, 3, \ldots$, for which the following statement is true.*

If X is a finite-dimensional vector space over $\mathbb{R}$ with $\dim(X) = m \geq 1$ and $\|\cdot\|$ a norm (or a subadditive positively homogeneous function) on X, then for a natural number $n > 1$ and for elements $x_j \in X$, $j = 1, \ldots, n$ with

$$\sum_{j=1}^{n} x_j = 0\,,$$

there exists a permutation $\sigma : \mathbb{N}_n \to \mathbb{N}_n$, such that $\sigma(1) = 1$, and

$$\|\sum_{j=1}^{k} x_{\sigma(j)}\| \leq C_m \max_{j \in \mathbb{N}_n} \|x_j\|, \; k = 1, 2, \ldots, n\,. \tag{14}$$

A proof of Theorem 13 with $C_m = 5^{\frac{m-1}{2}}$, $m = 1, 2, \ldots$ is indicated in (Ch. VII, Section 3, Exercise 1b, [57]). A version of Theorem 13 with $C_m = 2^m - 1$, $m = 1, 2, \ldots$ (without proof and with references to [23,52]) was formulated as Lemma 1 in [49].

It is remarkable that Theorem 13 holds for every norm given on a vector space X over $\mathbb{R}$ with $\dim(X) = m \geq 1$. For a fixed m (and concrete norm), the optimal value of the constant in (14), as well as an elaboration of an optimal algorithm for finding the corresponding permutation σ, plays a role in scheduling theory [58]. It is conjectured that in the case of Euclidean norms, the theorem should hold with constants $C_m, m = 1, 2, \ldots$ for which the sequence $\frac{C_m}{\sqrt{m}}, m = 1, 2, \ldots$ is bounded [59].

Using Theorem 13, it is possible to prove the following generalization of Theorem 7:

Theorem 14 (see Lemma 2, [49], and the rearrangement theorem (p. 346), [48])**.** *Let X be a finite-dimensional normed space over $\mathbb{R}$ and $\mathbf{x} = (x_n)_{n \in \mathbb{N}}$ be a sequence of elements of X, such that* $\mathrm{LPR}[\mathbf{x}; X] \neq \emptyset$. *The following statements are equivalent:*

(i) $\lim_n x_n = 0\,.$

(ii) $\mathrm{SR}[\mathbf{x}; X] = \mathrm{LPR}[\mathbf{x}; X]$.

It is known that if for a Banach space X over $\mathbb{R}$, an analogue (of implication $(i) \implies (ii)$) of Theorem 14 is true, then X is finite-dimensional [60].

The following assertion implies in particular that an analogue of Theorem 12$(I1)$ is not true for all nuclear spaces.

Theorem 15 (Kadets, [43])**.** *For an infinite-dimensional complete separable metrizable topological vector space X over $\mathbb{R}$, the following statements are equivalent.*

(SI) *For each sequence $\mathbf{x} = (x_n)_{n \in \mathbb{N}}$ in X, such that*

$$\mathrm{SR}[(x^*(x_n))_{n \in \mathbb{N}}; \mathbb{R}] \neq \emptyset \;\forall x^* \in X^*,$$

we have that $\mathrm{SR}[\mathbf{x}; X] \neq \emptyset$.

(Is) *X is topologically isomorphic to $\mathbb{R}^{\mathbb{N}}$ endowed with the topology of coordinatewise convergence.*

The following example shows that a further improvement of Theorem 12$(I1)$ is not possible even in the two-dimensional case.

Example 3 (Kadets, [43])**.** *Let X be a two-dimensional normed vector space over $\mathbb{R}$. There is a sequence $\mathbf{x} = (x_n)_{n \in \mathbb{N}}$ in X, such that the set*

$$K_{\mathbf{x}} := \{x^* \in X^* : \mathrm{SR}[(x^*(x_n))_{n \in \mathbb{N}}; \mathbb{R}] \neq \emptyset\}$$

separates points of X, but the set

$$\mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X]$$

is empty.

Proof. Take $X = \mathbb{R}^2$. Fix $n \in \mathbb{N}$, put

$$a_n = \frac{1+2(-1)^n}{n}, \ b_n = \frac{1-2(-1)^n}{n}, \ x_n = (a_n, b_n),$$

and consider the sequences $\mathbf{a} = (a_n)_{n\in\mathbb{N}}$, $\mathbf{b} = (b_n)_{n\in\mathbb{N}}$, and $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$. Then,

(1) $\mathrm{SR}[(a_n)_{n\in\mathbb{N}}; \mathbb{R}] = \mathbb{R}$, and $\mathrm{SR}[(b_n)_{n\in\mathbb{N}}; \mathbb{R}] = \mathbb{R}$.
(2) $\mathbb{R}e_1^* \cup \mathbb{R}e_2^* \subset K_{\mathbf{x}}$, where e_1^* and e_2^* are the first and the second projections from $X = \mathbb{R}^2$ onto $\mathbb{R}$, respectively. In particular, $K_{\mathbf{x}}$ separates points of $X = \mathbb{R}^2$.
(3) $\mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X] = \emptyset$.

(1) This follows from implication $(B) \Longrightarrow (D)$ in Theorem 5.
(2) This follows from (1).
(3) Suppose (3) is not true, i.e., $\mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X] \neq \emptyset$. Then, we can find and fix a permutation $\sigma : \mathbb{N} \to \mathbb{N}$, for which the series corresponding to $(x_{\sigma(n)})_{n\in\mathbb{N}}$ converges in $X = \mathbb{R}^2$. Then, both series corresponding to $(a_{\sigma(n)})_{n\in\mathbb{N}}$ and to $(b_{\sigma(n)})_{n\in\mathbb{N}}$ will converge in $\mathbb{R}$. This would imply that the series corresponding to $(a_{\sigma(n)} + b_{\sigma(n)})_{n\in\mathbb{N}}$ will converge in $\mathbb{R}$ too; but this is impossible, as $a_{\sigma(n)} + b_{\sigma(n)} = \frac{2}{\sigma(n)}, \ n = 1, 2, \ldots$.

□

4. Kadets-Type Theorems

The first result for infinite-dimensional Banach spaces was obtained in 1953 by Mikhail Iosifovich Kadets (1923–2011).

Theorem 16 (Kadets, Lemma I and Theorem II, [61])**.** *Let $1 < p < \infty$, (T, Σ, ν) be some σ-finite positive measure space, $X = \big(L_p(T, \Sigma, \nu; \mathbb{R}), \|\cdot\|_p\big)$, and $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ be a sequence in X. Then,*

$$\mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X] = \mathrm{LPR}[(x_n)_{n\in\mathbb{N}}; X],$$

and the sum range

$$\mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X]$$

is a closed affine subset of X, provided the following condition is satisfied:

(KC_p) $\sum_{n=1}^{\infty} \|x_n\|_p^{\min(2,p)} < +\infty$.

At the end of [61], it was written: "It is unknown for the author whether the condition (KC_p) is necessary".

Then, the following more general result appeared:

Theorem 17. (Stanimir Troyanski, [62]) *Let X be a uniformly smooth Banach space over $\mathbb{R}$ with a modulus of smoothness ρ. Then, for a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ in X, the sum range*

$$\mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X]$$

is a closed affine subset of X provided the following condition is satisfied:

$$(TC) \sum_{n=1}^{\infty} \rho(\|x_n\|) < +\infty.$$

In [62], it is noted also that it is unknown whether the condition (TC) is necessary.

Remark 7. *Ref. [62] began with the following definition: "A series of vectors of linear topological space is called conditionally convergent if two of its permutations have different sums." In view* *of this definition, one can expect that for a sequence* $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ *in Hausdorff topological vector space* X*, the following statement must be true:*

(HM) *If* $\mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X]$ *is a singleton, then the series corresponding to* $\mathbf{x}$ *is unconditionally convergent in* X*.*

We have:

- *(a)* *If* X *is finite-dimensional, then* (HM) *is true.*
  This follows from Steinitz's Theorem 12$(I3)$*.*
- *(b)* *If* X *is an infinite dimensional Hilbert space, then* (HM) *is not true [63].*
- *(c)* *If* X *is an infinite dimensional Banach space, then* (HM) *is not true either [64].*
- *(d)* *If* $X = \mathbb{R}^{[0,1]}$ *is endowed with the topology of point-wise convergence, then* (HM) *is not true, and there even exists a sequence* $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ *in* X *consisting of continuous functions, such that* $\mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X] = \{0\}$*, and* $\mathrm{SR}[(|x_n(t)|)_{n\in\mathbb{N}}; \mathbb{R}] = \emptyset$*, for each* $t \in [0,1]$ *[65].*

Next was paper [66], which was the first one to be written in English dealing with infinite-dimensional Hilbert spaces (and containing a conclusion about the sum range in the line of Steinitz's Theorem 12$(I3)$).

Theorem 18 (Vladimir Drobot, Theorem 1, [66])**.** *Let* (T, Σ, ν) *be* $[0,1]$ *endowed with the sigma-algebra* Σ *of Lebesgue-measurable sets and the Lebesgue measure* ν*,* $X = (L_2(T, \Sigma, \nu; \mathbb{R}), \|\cdot\|_2)$ *be the Hilbert space over* $\mathbb{R}$*, and* $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ *be a sequence in* X *with the following properties:*

- *(a)* $\mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X] \neq \emptyset$.
- *(b)* $\sum_{n=1}^{\infty} \|x_n\|_2 = +\infty$.
- *(c)* $\sum_{n=1}^{\infty} \|x_n\|_2^2 < +\infty$.
- *(d)* *The set* $\Gamma_{\mathbf{x}}$ *is closed in* $X^* = X$*.*

Then,

$$\mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X] = s + {}^{\perp}\Gamma_{\mathbf{x}},$$

for some $s \in \mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X]$*.*

Among all the previous works, ref. [66] only mentions Steinitz's 1913 paper. At the end of [66], two examples are presented.

Example 1 demonstrates that, in Theorem 18, conditions (a, b, c) may be satisfied, while condition (d) is not;

Example 2 gives a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ in X for which the conditions (a, b, c, d) of Theorem 18 with $\Gamma_{\mathbf{x}} = \{0\}$ are satisfied; hence, by this theorem, we have that $\mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X] = X$.

The last example, together with a version of Theorem 18 in which $\Gamma_{\mathbf{x}} = \{0\}$, is included in the monograph [67].

It is a bit strange that in [66] the result is not formulated or proved for an abstract infinite-dimensional Hilbert space over $\mathbb{R}$, while later in [68], one of the main ingredients of its proof is formulated and proved for the abstract case.

The first paper, written in Russian, to mention [66] was [69].

Theorem 19 (Vladimir Fonf, Theorem, [69])**.** *Let* X *be a uniformly smooth Banach space over* $\mathbb{R}$ *with a modulus of smoothness* ρ *and* $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ *a sequence in* X*, such that*

- *(a)* $\mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X] \neq \emptyset$.
- *(b)* $\sum_{n=1}^{\infty} \|x_n\| = +\infty$.
- *(c)* $\sum_{n=1}^{\infty} \rho(\|x_n\|) < +\infty$.

Then,

$$\mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X] = s + {}^{\perp}\Gamma_{\mathbf{x}}$$

for some $s \in \mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X]$*.*

Note that this theorem contains and improves Theorem 18, removing condition (d) from it.

In 1971, a paper [70] by Evgenii Mikhailovich Nikishin (1945-1986) appeared, where among other results, it was shown that in Kadets' Theorem 16, condition (KC_p) is in a sense the best possible when $p \geq 2$, and the above considered Banach's question from the "Scottish book" was answered negatively without actually knowing it.

Theorem 20 (Nikishin, Corollary 4 (p. 284), [70]). *Let $1 \leq p < \infty$. Let (T, Σ, ν) be $[-1, 1]$ endowed with the sigma-algebra Σ of Lebesgue-measurable sets and the Lebesgue measure ν $X = \big(L_p(T, \Sigma, \nu; \mathbb{R}), \|\cdot\|_p\big)$ Then, there exists a sequence $\varphi_n \in X,\ n = 1, 2, \ldots$, such that*

$$\sum_{n=1}^{\infty} \|\varphi_n\|_p^{2+\varepsilon} < \infty \qquad \forall \varepsilon > 0\,,$$

$\mathrm{SR}[(\varphi_n)_{n\in\mathbb{N}}; X] \neq \emptyset$, and $\mathrm{SR}[(\varphi_n)_{n\in\mathbb{N}}; X]$ is not affine.

Now, we formulate several other remarkable results contained in [70,71].

Theorem 21. *Let ν be the Lebesgue measure on $[0, 1]$ and $X = L_0$ the vector space over $\mathbb{R}$ of (ν-equivalence classes of) all ν-measurable functions $\varphi : [0, 1] \to \mathbb{R}$; moreover, let X_ν be the space X endowed with the topology of convergence in measure ν and $X_{\nu,a.e}$ be the space X endowed with ν-almost everywhere convergence (sequences from X). For a sequence $\varphi_n \in X,\ n = 1, 2, \ldots$, we write:*

$$\mathrm{SR}_\nu[(\varphi_n)_{n\in\mathbb{N}}] := \mathrm{SR}[(\varphi_n)_{n\in\mathbb{N}}]; X_\nu],$$

and

$$\mathrm{SR}_{\nu,a.e}[(\varphi_n)_{n\in\mathbb{N}}] := \mathrm{SR}[(\varphi_n)_{n\in\mathbb{N}}]; X_{\nu,a.e}]\,.$$

The following statements are valid.

(I) (Theorem 1, [71]) (see Theorem 7) If for a sequence $\psi_n \in X,\ n = 1, 2, \ldots$,
- *(a) The series corresponding to the sequence $(\psi_n^2)_{n\in\mathbb{N}}$ converges in $\mathbb{R}$ ν-almost everywhere, and*
- *(b) Some subsequence of the sequence $\sum_{k=1}^{n} \psi_k \in X,\ n = 1, 2, \ldots$ converges ν-almost everywhere to a function $\varphi \in X$,*

then $\varphi \in \mathrm{SR}_{\nu,a.e}[(\psi_n)_{n\in\mathbb{N}}]$.

(II) (Theorem 2, [71]) If for a sequence $\varphi_n \in X,\ n = 1, 2, \ldots$, the series corresponding to the sequence $(\varphi_n^2)_{n\in\mathbb{N}}$ converges in $\mathbb{R}$ ν-almost everywhere, then

(IIa) (Theorem 2, [71]) $\mathrm{SR}_{\nu,a.e}[(\varphi_n)_{n\in\mathbb{N}}]$ is an affine subset of X, which is closed in X_ν.

(IIb) $\mathrm{SR}_{\nu,a.e}[(\varphi_n)_{n\in\mathbb{N}}] = \mathrm{SR}_\nu[(\varphi_n)_{n\in\mathbb{N}}]$.

(IIc) $\mathrm{SR}_\nu[(\varphi_n)_{n\in\mathbb{N}}]$ is a closed affine subset of X_ν.

(III) Ref. [70] There exists a sequence $\varphi_n \in X,\ n = 1, 2, \ldots$, such that the series corresponding to the sequence $(|\varphi_n|^{2+\varepsilon})_{n\in\mathbb{N}}$ converges in $\mathbb{R}$ ν-almost everywhere for every $\varepsilon > 0$, and $\mathrm{SR}_{\nu,a.e}[(\varphi_n)_{n\in\mathbb{N}}]$ is not an affine subset of X.

Comments on the Proof. (IIb) The inclusion $\mathrm{SR}_{\nu,a.e}[(\varphi_n)_{n\in\mathbb{N}}] \subset \mathrm{SR}_\nu[(\varphi_n)_{n\in\mathbb{N}}]$ is true because the convergence of sequences ν-almost everywhere implies the convergence in measure.

The proof of the inclusion $\mathrm{SR}_\nu[(\varphi_n)_{n\in\mathbb{N}}] \subset \mathrm{SR}_{\nu,a.e}[(\varphi_n)_{n\in\mathbb{N}}]$ is more delicate. So, we fix $\varphi \in \mathrm{SR}_\nu[(\varphi_n)_{n\in\mathbb{N}}]$ and take a permutation $\sigma : \mathbb{N} \to \mathbb{N}$ for which the series corresponding to the sequence $(\varphi_{\sigma(n)})_{n\in\mathbb{N}}$ converges in measure to φ. Then, it is easy to see that the assumptions (a) and (b) of (I) are satisfied for the sequence $\psi_n := \varphi_{\sigma(n)},\ n = 1, 2, \ldots$, and from (I), we conclude that

$$\varphi \in \mathrm{SR}_{\nu,a.e}[(\psi_n)_{n\in\mathbb{N}}] = \mathrm{SR}_{\nu,a.e}[(\varphi_n)_{n\in\mathbb{N}}]\,.$$

(IIc) follows from (IIb) and (IIa). $\square$

As noted in [70], Theorem 21(III) answers negatively, in particular, a question which (according to [72]) was posed by Banach. The following assertion shows that in Kadets' Theorem 16 the condition (KC_p) is in a sense the best possible one also when $1 < p < 2$.

Theorem 22 (Kornilov, see Theorem 1, [73])**.** *Let $1 \le p \le 2$, (T, Σ, ν) be $[0,1]$ endowed with the sigma-algebra Σ of Lebesgue-measurable sets and the Lebesgue measure ν, $X = \big(L_p(T, \Sigma, \nu; \mathbb{R}), \|\cdot\|_p\big)$ and let $\omega :]0, \infty[\to]0, \infty[$ be a function, such that*

$$\lim_{t \to 0} \omega(t) = 0\,.$$

Then, there exists a sequence $\varphi_n \in X,\ n = 1, 2, \ldots,$ such that

(1) $\sum_{n=1}^{\infty} \|\varphi_n\|_p^p \omega(\|\varphi_n\|_p) < \infty$,

(2) $0 \in \mathrm{SR}[(\varphi_n)_{n\in\mathbb{N}}; X]$ *and* $1 \in \mathrm{SR}[(\varphi_n)_{n\in\mathbb{N}}; X]$,
but

(3) $\lambda \in]0, 1[\implies \lambda \notin \mathrm{SR}[(\varphi_n)_{n\in\mathbb{N}}; X]$.

In 1973, the following variant of Theorem 16 was announced, which can be considered the first infinite-dimensional version of Steinitz's Theorem 12$(I2)$:

Theorem 23. *Let $1 < p < \infty$, (T, Σ, ν) be $[0,1]$ endowed with the σ-algebra Σ of Lebesgue-measurable sets, and the Lebesgue measure ν and $X = \big(L_p(T, \Sigma, \nu; \mathbb{R}), \|\cdot\|_p\big)$. Let, moreover, $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ be a sequence in X, which satisfies the condition (KC_p).*

Then, the following statements are valid.

(I) *(Pecherskii, (Theorem 1, [74])) The equality*

$$\mathrm{StR}[(x_n)_{n\in\mathbb{N}}; X] = \mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X]$$

holds.

(II) *Pecherskii, (Theorem 3, [74]))*

$$\forall x^* \in X^* \setminus \{0\} \qquad \mathrm{SR}[(x^*(x_n))_{n\in\mathbb{N}}; \mathbb{R}] = \mathbb{R} \implies \mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X] = X\,.$$

We note that Theorem 23(II), in the case when $p = 2$, is included with a complete proof in (Appendix, Section 6, Theorem 1 (pp. 352–357), [75]).

The first essential improvement of Theorem 16 in case $1 < p < 2$ was the following result.

Theorem 24 (Nikishin, Theorem 1, [76])**.** *Let $1 \le p < 2$, (T, Σ, ν) be $[0,1]$ endowed with the sigma-algebra Σ of Lebesgue-measurable sets and the Lebesgue measure ν, and $X = \big(L_p(T, \Sigma, \nu; \mathbb{R}), \|\cdot\|_p\big)$. Moreover, let $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ be a sequence in X that satisfies the following condition.*

$(NikC_p)$, the series corresponding to the sequence $(|x_n(t)|^2)_{n\in\mathbb{N}}$, converges in $\mathbb{R}$ for Lebesgue's almost every $t \in [0,1]$ and

$$\left(\sum_{n=1}^{\infty} |x_n|^2\right)^{\frac{1}{2}} \in X\,.$$

Then, for the sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$, the sum range

$$\mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X]$$

is a closed affine subset of X.

In 1977, the following modification Theorem 23 appeared, which takes into account Nikishin's Theorem 24 too.

Theorem 25. *Let $1 \le p < \infty$, (T, Σ, ν) be $[0,1]$ endowed with the σ-algebra Σ of Lebesgue-measurable sets and the Lebesgue measure ν and $X = \big(L_p(T, \Sigma, \nu; \mathbb{R}), \|\cdot\|_p\big)$. Moreover, let*

$\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ *be a sequence in* X *for which* $(NikC_p)$ *is satisfied when* $1 \le p < 2$, *and* (KC_2) *is satisfied, when* $2 \le p < \infty$. *The following statements are true.*

(I) *(Theorem 1, [77]) The equality*

$$\mathrm{StR}[(x_n)_{n\in\mathbb{N}}; X] = \mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X]$$

holds.

(II) *(Corollary 1, [77]) If*

$$\mathrm{SR}[(x^*(x_n))_{n\in\mathbb{N}}; \mathbb{R}] = \mathbb{R} \quad \forall x^* \in X^* \setminus \{0\},$$

then the equality

$$\mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X] = X$$

holds too.

The following result is applicable to the non-separable Banach space $(l_\infty, \|\cdot\|_\infty)$ of all bounded real sequences.

Theorem 26 (Barany, Theorems 2 and 3, [78])**.** *Let* $1 \le p \le \infty$, $c(j) := 2^{3^j}$, $j = 1, 2, \ldots$, *and* $X = (l_p, \|\cdot\|_p)$. *Moreover, let* $x_n : \mathbb{N} \to \mathbb{R}$, $n = 1, 2, \ldots$ *be a sequence in* X *such that* $c \cdot x_n \in X$, $n = 1, 2, \ldots$.

Assume further that either

$(BaCo_\infty)$ $p = \infty$, *and* $\limsup_n \|c \cdot x_n\|_\infty = 0$,

or

$(BaCo_p)$ $1 \le p < \infty$, *and* $\sum_{n=1}^{\infty} \|c \cdot x_n\|_p^p < \infty$.

Then, for the sequence $(x_n)_{n\in\mathbb{N}}$, *the sum range,*

$$\mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X]$$

is a closed affine subset of X.

The result is new when $p = \infty$. In the case when $1 \le p \le 2$, it is a consequence of Kadets' Theorem 16, while in the case when $2 < p < \infty$, it is independent from this theorem.

From Theorem 26, unlike the previous results in the present section, it is possible to derive the following corollary.

Corollary 1 (see Proposition 5)**.** *Let* $(t_n)_{n\in\mathbb{N}}$ *be a sequence of real numbers. Then, the sum range*

$$\mathrm{SR}[(t_n)_{n\in\mathbb{N}}; \mathbb{R}]$$

can be either empty, a singleton, or $\mathbb{R}$.

Proof. If $\limsup_n |t_n| > 0$, then clearly $\mathrm{SR}[(t_n)_{n\in\mathbb{N}}; \mathbb{R}] = \varnothing$. So, let $\limsup_n |t_n| = 0$. Consider the sequence $x_n := t_n e_1 \in l_\infty$, $n = 1, 2, \ldots$, where $e_1 = (1, 0, 0, \ldots, 0, \ldots)$. Clearly, $c \cdot x_n = c_1 t_n e_1$, $n = 1, 2, \ldots$, and so, $\limsup_n \|c \cdot x_n\|_\infty = c_1 \limsup_n |t_n| = 0$. Hence, $(BaCo_\infty)$ is satisfied for our sequence $(x_n)_{n\in\mathbb{N}}$, and then by Theorem 26, $\mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X]$ is a closed affine subset of X. Clearly,

$$\mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X] = \mathrm{SR}[(t_n e_1)_{n\in\mathbb{N}}; X] \subset \mathbb{R} \cdot e_1 = \{t e_1 : t \in \mathbb{R}\}.$$

From this relation, we conclude that

$$\mathrm{SR}[(t_n)_{n\in\mathbb{N}}; \mathbb{R}]$$

is a closed affine subset of $\mathbb{R}$. □

Of course, it would be interesting to find other sequences $c(j)$, $j = 1, 2, \ldots$ for which Theorem 26 will remain true.

The first generalizations of Nikishin's Theorem 24 appeared in [79,80]. To formulate them, we recall the needed definitions.

For a natural number n, the Rademacher function $r_n : [0,1] \to \{-1,1\}$ is defined by the equality

$$r_n(t) = (-1)^{[2^n t]}, \qquad t \in [0,1],$$

where $[x]$ stands for the integer part of $x \in \mathbb{R}$.

We say that a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ in a Banach space X over $\mathbb{R}$ satisfies the (RSC)-condition, if for Lebesgue's almost every $t \in [0,1]$, the series corresponding to the sequence $(r_n(t)x_n)_{n\in\mathbb{N}}$ converges in X.

For a number $q \in \mathbb{R},\ q \geq 2$, we say that a Banach space X over $\mathbb{R}$ is of cotype q, if for every sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ in X satisfying the (RSC)-condition, the series corresponding to the sequence $(\|x_n\|^q)_{n\in\mathbb{N}}$ converges in $\mathbb{R}$.

Theorem 27 (Theorems 8(a,b) and 9, [79])**.** *Let X be a Banach space X over $\mathbb{R}$ and $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ a sequence in X, such that*

$$\mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X] \neq \emptyset\,.$$

If either

(I) $X = L_p(T, \Sigma, \nu; \mathbb{R})$, with $1 \leq p \leq 4$ and with some σ-finite positive measure space (T, Σ, ν), and for $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$, the condition $(NikC_p)$ is satisfied,
or

(II) X is of cotype 2, and the sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ satisfies the (RSC)-condition,
then

(III) The equality

$$\mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X] = \mathrm{StR}[(x_n)_{n\in\mathbb{N}}; X]$$

holds.

This theorem contains the promised first generalizations of Nikishin's Theorem 24. At the very end of [79], it was conjectured that $((II) \Longrightarrow (III))$ in Theorem 27 should be true for all Banach spaces. Soon, this conjecture was confirmed. See Theorem 30 below.

Theorem 28 (Particular cases of Theorems 1 and 2, [80])**.** *Let $0 \leq p < \infty$, (T, Σ, ν) be some σ-finite positive measure space, $X = L_p(T, \Sigma, \nu; \mathbb{R})$, and $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ be a sequence in X.*

(I) If for $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$, the condition $(NikC_p)$ is satisfied, then the sum range

$$\mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X]$$

is a closed affine subset of X.

(II) If $1 \leq p < \infty$, for $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$, the condition $(NikC_p)$ is satisfied, and $\mathrm{StR}[(x_n)_{n\in\mathbb{N}}; X] \neq \emptyset$, then the equality

$$\mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X] = s + {}^{\perp}\Gamma_{\mathbf{x}}$$

holds for each $s \in \mathrm{StR}[(x_n)_{n\in\mathbb{N}}; X]$.

This theorem contains the further generalizations of Nikishin's Theorem 24. Theorem 28(II) also extends Theorem 25(I).

For a number $r \in \mathbb{R}, 1 < r \leq 2$, we say that a Banach space X over $\mathbb{R}$ is

- Of *type* r, if for every sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ in X, for which the series corresponding to the sequence $(\|x_n\|^r)_{n\in\mathbb{N}}$ converges in $\mathbb{R}$, the (RSC)-condition is satisfied.
- Of *infratype* r, if there exists a positive finite constant C, such that for each natural number n and elements $x_k \in X$, $k = 1, \ldots, n$, the inequality

$$\|\sum_{k=1}^{n} \theta_k x_k\| \leq C \left(\sum_{k=1}^{n} \|x_k\|^r \right)^{\frac{1}{r}}$$

holds for some choice of 'signs' $\theta_k \in \{-1,1\}$, $k = 1,\ldots,n$.

In (Chapter 7, Section 1 (pp. 87–92) [5]), a machinery oriented to obtaining the following result is developed.

Theorem 29 (Kadets–Ostrovskii [35,81] and Theorem 7.1.2 (p. 92), [5])**.** *Let $1 < r \leq 2$, X be a Banach space over $\mathbb{R}$ having the infratype r, and $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ be a sequence in X for which*

(a) $\mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X] \neq \emptyset$, *and*

(b) *The sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ satisfies the condition:*
(KC_r) The series corresponding to the sequence $(\|x_n\|^r)_{n\in\mathbb{N}}$ converges in $\mathbb{R}$.
Then, the equality
$$\mathrm{SR}((x_n)_{n\in\mathbb{N}}) = s + {}^{\perp}\Gamma_{\mathbf{x}}$$
holds for each $s \in \mathrm{SR}((x_n)_{n\in\mathbb{N}})$.

Theorem 29 covers Theorem 16 in the case when $1 \leq p < \infty$, as if $X = \big(L_p([0,1]), \|\cdot\|_p\big)$, then it is known that X is of type $r = \min(p,2)$.

Theorem 30 (Announced in Theorem 3, [82], and proved in Theorem 5, [31])**.** *Let X be the Banach space $\mathbb{R}$, and let $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ be a sequence in X for which*

(a) $\mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X] \neq \emptyset$, *and*

(b) *The sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ satisfies the (RSC)-condition.*
Then, the equality
$$\mathrm{StR}[(x_n)_{n\in\mathbb{N}}; X] = \mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X]$$
holds.

This statement was the first general result valid for all Banach spaces. It does not cover Theorem 26, when $2 < p \leq \infty$. However, it implies the following final improvement of Nikishin's Theorem 24.

Theorem 31 (Announced in Corollary 2, [82], and proved in Corollary 3, [31])**.** *Let $X = L_p(T,\Sigma,\nu;\mathbb{R})$, with $1 \leq p < +\infty$ and with some σ-finite positive measure space (T,Σ,ν), and let $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ be a sequence in X for which*

(a) $\mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X] \neq \emptyset$, *and*

(b) *The series corresponding to the sequence $(|x_n(t)|^2)_{n\in\mathbb{N}}$ converges in $\mathbb{R}$ for ν-almost every $t \in T$, and*
$$\left(\sum_{n=1}^{\infty} |x_n|^2\right)^{\frac{1}{2}} \in X\,.$$
Then, the equality
$$\mathrm{StR}[(x_n)_{n\in\mathbb{N}}; X] = \mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X]$$
holds.

Comments on the Proof. This follows from Theorem 30 due to the following theorem by Jorgen Hoffman-Jorgensen (1942-2017): (b) *holds if and only if the sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ satisfies the (RSC)-condition* (Corollary 2(b) to Theorem 5.5.2 (pp. 323–324), [83]). □

Remark 8. *We fix $r \in]1,2]$ and a Banach space X over $\mathbb{R}$. Let us note:*

(1) *If X is of type r, then Theorem 29 follows from Theorem 30.*

(2) *If $1 < r < 2$, and X is of infratype r, then X is of type r too [84]. From this and (1), we conclude that if $1 < r < 2$, and X is of infratype r, then again Theorem 29 follows from Theorem 30.*

(3) *Ref. [85] showed the existence of X of infratype 2, which is not of type 2. Consequently, in the case of $p = 2$, Theorem 29 does not follow from Theorem 30.*

To formulate an important generalization of Theorem 30, it would be convenient to provide a definition.

We say that a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ in the Banach space (or the topological vector space) X over $\mathbb{R}$ satisfies the (PSC)-condition, if for every permutation $\sigma : \mathbb{N} \to \mathbb{N}$, there exists a sequence of 'signs' $\theta_n \in \{1, -1\}$, $n = 1, 2, \ldots$, such that the series corresponding to the sequence $(\theta_n x_{\sigma(n)})_{n\in\mathbb{N}}$ converges in X.

Theorem 32. *Let X be a Banach space over $\mathbb{R}$ and $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ be a sequence in X, which satisfies the (PSC)-condition.*

Then,

(I) *(Theorem 1, [86]) The equality*

$$\mathrm{StR}[(x_n)_{n\in\mathbb{N}}; X] = \mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X]$$

holds.

(II) *(Corollary 2, [86]) The equality*

$$\mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X] = X$$

holds, if and only if

$$\mathrm{SR}[(x^*(x_n))_{n\in\mathbb{N}}; \mathbb{R}] = \mathbb{R} \qquad \forall x^* \in X^* \setminus \{0\}\,. \tag{15}$$

In particular, if (15) is satisfied, then $\mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X] \neq \emptyset$.

Theorem 32(I) implies Theorem 30 because, as proved in (Proposition 1, [86]) in a somewhat sophisticated way, the (RSC)-condition implies the (PSC)-condition. Note also that Theorem 32 would imply Theorem 29 in the case when $p = 2$ too, if the following conjecture is true.

Conjecture 1 (Infratype 2 conjecture; see Conjecture (p. 92), [5])**.** *Let X be a Banach space of infratype 2. Then, for every sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ in X for which the series corresponding to the sequence $(\|x_n\|^2)_{n\in\mathbb{N}}$ converges in $\mathbb{R}$, there exists a sequence of 'signs' $\theta_n \in \{1, -1\}$, $n = 1, 2, \ldots$, such that the series corresponding to the sequence $(\theta_n x_n)_{n\in\mathbb{N}}$ converges in X.*

The following, weaker version, of Theorem 32(I) was announced (independently of [86]) in [87] and is included in [5] as "Pecherskii's theorem".

Theorem 33 (Announced in Theorem 4, [87], and proved in Theorem 2.3.1, [5])**.** *Let X be a Banach space over $\mathbb{R}$ and $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ be a sequence in X for which*

(a) $\mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X] \neq \emptyset$, *and*

(b) *The sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ satisfies the (PSC)-condition.*

Then, the equality

$$\mathrm{StR}[(x_n)_{n\in\mathbb{N}}; X] = \mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X]$$

holds.

In (p. 23, [5]), the following observation is included after the formulation of Theorem 33:

(1) "This assertion provides the most general of the known sufficient conditions for linearity of the sum range of a series in an infinite-dimensional space".

(2) "In the finite-dimensional case Theorem 2.3.1 is identical to Steinitz's theorem..."

  (1) This is not completely true, as above, we state here too: Theorem 33 would imply Theorem 29 in the case when $p = 2$, if the infratype 2 conjecture were true.

  (2) This is true due to the following result.

Theorem 34 (Dvoretzky–Hanani theorem, [19] in the case when $\dim(X) = 2$ and Theorem 2.2.1 (p. 22), [5], in general)**.** *Let X be a finite-dimensional normed space over $\mathbb{R}$ and $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ be a sequence in X, which converges to zero in X. Then, there exists a sequence of 'signs' $\theta_n \in \{1, -1\}$, $n = 1, 2, \ldots$, such that the series corresponding to the sequence $(\theta_n x_n)_{n\in\mathbb{N}}$ converges in X.*

We note that this result is presented on p. 24 of the Russian edition of [44] as Exercise 1.3.7; then, on p. 28 after Exercise 2.1.2, it is noted that it is equivalent to the following theorem.

Theorem 35 ([19] in the case when $\dim(X) = 2$ and Lemma 2.2.1 (p. 21), [5], in general; see also [59])**.** *There exists a sequence $D_m, m = 1, 2, \ldots$ of strictly positive constants, with $D_1 = 1$ and with $D_m \leq 2m - 1, m = 2, 3, \ldots$, for which the following statement is true.*

Let X be a finite-dimensional vector space over $\mathbb{R}$ with $\dim(X) = m \geq 1$, and let $\|\cdot\|$ be a norm on X. Then, for a natural number $n > 1$ and for elements $x_j \in X$, $j = 1, \ldots, n$, there exist 'signs' $\theta_j \in \{-1, 1\}$, $j = 1, \ldots, n$, such that

$$\| \sum_{j=1}^{k} \theta_j x_j \| \leq D_m \max_{j\in\mathbb{N}_n} \|x_j\|, \; k = 1, 2, \ldots, n\,. \tag{16}$$

The following version of Theorem 35 (without proof and with a reference to [88]) was formulated as Lemma 10 in [49].

There exists a sequence $D_m, m = 1, 2, \ldots$ of strictly positive constants for which the following statement is true.

If $X = \mathbb{R}^m$ with $m \in \mathbb{N}$ and with the maximum norm $\|\cdot\|$ on X, then for a bounded sequence of elements $x_j \in X$, $j = 1, 2, \ldots$, there exist 'signs' $\theta_j \in \{-1, 1\}$, $j = 1, 2, \ldots$, such that

$$\| \sum_{j=1}^{k} \theta_j x_j \| \leq D_m \sup_{j\in\mathbb{N}} \|x_j\|, \; k = 1, 2, \ldots\,. \tag{17}$$

However, in [88] it is hard to find such a statement. It seems that in the case when $m > 2$, the first proof of Theorem 35 is Grinberg's proof, which appeared on pp. 178–179 of the Russian edition of [44] as a solution to Exercise 2.1.2.

It is conjectured (as in the case of Theorem 13) that for Euclidean norms, the theorem should hold with constants $D_m, m = 1, 2, \ldots$ for which the sequence $\frac{D_m}{\sqrt{m}}, m = 1, 2, \ldots$ is bounded [59].

The following result covers some cases of metrizable topological vector spaces, which may not be locally convex.

Theorem 36 (Giorgobiani)**.** *Let X be a metrizable topological vector space over $\mathbb{R}$, and let $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ be a sequence in X, which satisfies the (PSC)-condition.*

Assume further that

$(GiCo)$ the topology of X can be generated by a translation invariant metric d, such that

$$\inf\left\{ \frac{d(2x, 0)}{d(x, 0)} : x \in X \setminus \{0\} \right\} > 1\,.$$

Then, the following statements are valid.

(a) *(Theorem 1.2.1 (p. 34), [39])* $\mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X] = \mathrm{LPR}[(x_n)_{n\in\mathbb{N}}; X]$.

(b) *(Announced in (Remark (p. 45), [87]), and proved in [89]; see also (Theorem 1.3.1 (p. 41), [39])* $\mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X]$ *is a closed affine subset of X.*

This theorem covers Theorem $28(I)$. Does Theorem 36 remain true for all metrizable topological vector spaces? The answer is unknown yet. The following result covers the general case of metrizable locally convex topological vector spaces.

Theorem 37 (Maria-Jesus Chasco–Sergei Chobanyan)**.** *Let X be a metrizable locally convex topological vector space over $\mathbb{R}$ and $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ be a sequence in X, which satisfies the (PSC)-condition. Then, the following statements are valid.*

(a) *(Theorem 2, [90])* $\mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X] = \mathrm{LPR}[(x_n)_{n\in\mathbb{N}}; X]$.

(b) *(Announced in (Theorem 5 (p. 15), [91]), also in [92], and proved in (Theorem 3, [90]); see also (Theorem 1.3.2 (p. 45), [39])*
If $\mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X] \neq \emptyset$*, then the equality*

$$\mathrm{StR}[(x_n)_{n\in\mathbb{N}}; X] = \mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X]$$

holds.

The following inequality plays a key role in the proof of Theorem 37.

Proposition 11 (Lemma 1, [90])**.** *Let $n \geq 2$ be a natural number, let X be a vector space over $\mathbb{R}$, $\|\cdot\|$ be a seminorm on X, and $a_k \in X$, $k = 1, 2, \ldots, n$. Moreover, let*

(1) *$\pi : \mathbb{N}_n \to \mathbb{N}_n$ be an 'optimal' permutation in the following sense: for any permutation $\lambda : \mathbb{N}_n \to \mathbb{N}_n$, the inequality*

$$\max_{1\leq k\leq n} \|\sum_{j=1}^{k} a_{\pi(j)}\| \leq \max_{1\leq k\leq n} \|\sum_{j=1}^{k} a_{\lambda(j)}\|$$

holds, and

(2) *$\sigma : \mathbb{N}_n \to \mathbb{N}_n$ be the permutation associated with π as follows*

$$\sigma(k) = \pi(n-k+1),\ k = 1, 2, \ldots, n$$

Then,

$$\max_{1\leq k\leq n} \|\sum_{j=1}^{k} a_{\pi(j)}\| \leq \|\sum_{j=1}^{n} a_j\| + \max_{1\leq k\leq n} \|\sum_{j=1}^{k} \theta_j a_{\sigma(j)}\|$$

for every choice of 'signs' $\theta_j \in \{1, -1\}$, $j = 1, 2, \ldots, n$.

Theorem 37 would imply Banaszczyk's Theorem 10, if the following conjecture is true.

Conjecture 2 ((p. 109), [6], and (p. 615), [90])**.** *Let X be a complete metrizable nuclear locally convex topological vector space over $\mathbb{R}$ and $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ be a sequence in X, which converges to zero in X. Then, there exists a sequence of 'signs' $\theta_n \in \{1, -1\}$, $n = 1, 2, \ldots$, such that the series corresponding to the sequence $(\theta_n x_n)_{n\in\mathbb{N}}$ converges in X.*

Conjecture 2 is true when X is finite dimensional by Theorem 34, when $X = \mathbb{R}^{\mathbb{N}}$ (Theorem 2, [49]), and for some other nuclear spaces [93]. The following result, related to this conjecture, is true.

Theorem 38 (Wojciech Banaszczyk, [94]; announced in (Remark 10.15 (pp. 106–107), [6]))**.** *For a complete metrizable locally convex topological vector space X over $\mathbb{R}$, the following statements are equivalent:*

(i) *X is nuclear.*

(ii) *For every sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ of elements of X, which converges to zero in X, there exists a sequence of 'signs' $\theta_n \in \{1, -1\}$, $n = 1, 2, \ldots$ and a permutation $\sigma : \mathbb{N} \to \mathbb{N}$, such that the series corresponding to the sequence $(\theta_n x_{\sigma(n)})_{n\in\mathbb{N}}$ converges in X.*

After [90], a remarkable paper by Bonet and Defant [95] and a paper by Sofi [96] appeared. The first one deals with Banaszczyk's type rearrangement theorems for (not necessarily metrizable) nuclear locally convex spaces. The second one contains Chasco–Chobanyan-type results imposing conditions on series different from the (PSC)-condition.

5. Additional Comments

During our expositions, we have indicated several problems, which would be interesting to solve. In connection with Chasco–Chobanyan's theorem, it would be interesting to answer also the following questions.

Question 4. *Is Theorem 37(b) true without the assumption that* $\mathrm{SR}[(x_n)_{n\in\mathbb{N}}; X] \neq \emptyset$*?*

Question 5. *Let X be as in Theorem 37 and $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ and $\mathbf{y} = (y_n)_{n\in\mathbb{N}}$ be two sequences in X, which satisfy the (PSC)-condition. Does their sum $(x_n + y_n)_{n\in\mathbb{N}}$ satisfy the (PSC)-condition?*

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