New results on egalitarian values for games with a priori unions

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Abstract

Several extensions of the equal division value and the equal surplus division value to the family of games with a priori unions have been proposed by Alonso-Meijide et al. (2020) in "On egalitarian values for cooperative games with a priori unions" TOP 28: 672-688. In this study, we provide new axiomatic characterizations of these values. Furthermore, using the equal surplus division value in two steps, we propose a new coalitional value. The balanced contributions and quotient game properties generate a different modification of the equal surplus division value.

Keywords: cooperative games, coalitional values, equal division value, equal surplus division value.

1 Introduction

One of the main subjects of study in the cooperative game theory is how to divide an existing amount among a set of agents. The Shapley value (Shapley, 1953) is arguably the most successful answer to this problem. In this study, we analyze two different alternatives to the Shapley value: the egalitarian solution (where the worth of the grand coalition is divided equally among the

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players), and the equal surplus division value (Driessen and Funaki, 1991), which first allocates individual payoffs to each agent and subsequently divides the remaining amount equally among them. These values satisfy several good properties; van den Brink and Funaki (2009), Chun and Park (2012), van den Brink et al. (2016), Ferrières (2017), and Bèal et al. (2019), among others, provide many axiomatic characterizations.

Players with similar interests are more likely to act together, giving rise to games where cooperation is restricted by an a priori system of unions. Different coalitional values have been analyzed for this type of game. The first one, proposed by Aumann and Drèze (1974), considers that every player receives the Shapley value of the game played within his union. Owen (1977) defined a different coalitional value (Owen value) based on the following process. First, the unions play a game among themselves (quotient game), and each union receives a payoff that is shared among its players in a second (internal) game. In both cases, the Shapley value is used to compute the corresponding payoffs. The Owen value coincides with the Shapley value when all unions are singletons; that is, it is a coalitional Shapley value. The Owen value satisfies several good properties, such as the quotient game and balanced contribution properties. The quotient game property states that the players of an a priori union receive the amount that this union receives in the quotient game. The balanced contributions property compares the amount obtained by two players of the same union in the original situation where one of them leaves this union. In Vázquez-Brage et al. (1997), the Owen value is characterized as the unique coalitional Shapley value satisfying the quotient game and balanced contributions properties.

Other values are extended to cooperative games with a priori unions. In the context that concerns us, Alonso-Meijide et al. (2020) extend and characterize the equal division value and the equal surplus division value. For the second value, three alternative ways to adapt it to a priori unions are proposed.

This study aims to provide a new axiomatic characterization for the values proposed by Alonso-Meijide et al. (2020), which can be compared with the characterization of the Owen value proposed by Vázquez-Brage et al. (1997). However, none of the extensions of the equal surplus division value satisfy both properties (quotient game and balanced contributions). Therefore, we propose variants of the properties that allow us to characterize the extensions proposed by Alonso-Meijide et al. (2020).

Further, we obtain a new modification of the equal surplus division value following the Owen process using the equal surplus division value (instead of the Shapley value) in two steps. Finally, we provide the expression of the coalitional extension of the equal surplus division value satisfying the same properties as those used by Vzquez-Brage et al. (1997) to characterize the Owen value.

2 Preliminaries

2.1 TU-games and values

A transferable utility cooperative game (from now on a TU-game) is a pair (N, v), where N is a finite set of n players, and v is a map from 2^N to $\mathbb{R}$ with $v(\emptyset) = 0$, called the games characteristic function. In the sequel, $\mathcal{G}_N$ denotes the family of all TU-games with player set N and $\mathcal{G}$ the family of all TU-games. A value for TU-games is a map f that assigns to every TU-game $(N, v) \in \mathcal{G}$ a vector $f(N, v) = (f_i(N, v))_{i \in N} \in \mathbb{R}^N$.

Well-known TU-game values are egalitarian values. The equal division value ED distributes $v(N)$ equally among the players in N. Formally, the equal division value ED is defined for every $(N, v) \in \mathcal{G}$, and every $i \in N$ is defined by

$$ED_i(N, v) = \frac{v(N)}{n}.$$

The equal surplus division value ESD is defined for every $(N, v) \in \mathcal{G}$, and every $i \in N$ by

$$ESD_i(N, v) = v(i) + \frac{v^0(N)}{n}$$

where $v^0(N) = v(N) - \sum_{i \in N} v(i)$. Notice that ESD is a variant of ED, where we first allocate $v(i)$ to each player i and then distribute $v^0(N)$ among the players using ED. ESD is a reasonable alternative to ED when individual benefits and joint benefits are neatly separable.

Alternative values for TU-games are the Shapley (Shapley 1953) and Banzhaf values (Banzhaf 1964).

2.2 Games with a priori unions

We denote the set of all partitions of N by $P(N)$. Then, a TU-game with a priori unions is a triplet (N, v, P), where $(N, v) \in \mathcal{G}$, $P = \{P_1, \ldots, P_m\} \in$

$P(N)$, and $P_k \in P$ is called a priori union for all $k \in M$ with $M = \{1, ..., m\}$. The set of TU-games with a priori unions and a player set N is denoted by $\mathcal{G}_N^U$, and the set of all TU-games with a priori unions is denoted by $\mathcal{G}^U$. A value for TU-games with a priori unions is a map g that assigns to every $(N, v, P) \in \mathcal{G}^U$ a vector $g(N, v, P) = (g_i(N, v, P))_{i \in N} \in \mathbb{R}^N$.

Two examples of values for TU-games with a priori unions are the Owen value (Owen 1977) and the Banzhaf-Owen value (Owen 1981).

Given $(N, v, P) \in \mathcal{G}^U$ with $P = \{P_1, \ldots, P_m\} \in P(N)$, the quotient game of (N, v, P) is the TU-game $(M, v/P)$, where

$$(v/P)(R) = v\left(\cup_{r \in R} P_r\right) \text{ for all } R \subseteq M.$$

We say that a value for TU-games with a priori unions g is a *coalitional equal division value (CED)* if, for any TU-game $(N, v) \in \mathcal{G}$, it holds that

$$g(N, v, P^n) = ED(N, v),$$

where

$$P^n = \{\{1\}, \ldots, \{n\}\}.$$

Using similar concepts, the Owen value is a coalitional Shapley value, and Banzhaf-Owen value is a coalitional Banzhaf value.

2.3 Coalitional values in two steps

Given $(N, v, P) \in \mathcal{G}_N^U$ with $P = \{P_1, \ldots, P_m\}$ and a coalition $S \subseteq P_r$, the modified game of (N, v, P) is defined as $(M, u_{r,S})$, where

$$u_{r,S}(H) = \begin{cases} v\left(\cup_{k \in H} P_k\right) & \text{if } r \notin H \\ v\left(\cup_{k \in H \setminus r} P_k \cup S\right) & \text{if } r \in H \end{cases} \tag{1}$$

For all $H \subseteq M$. That is, the modified game $(M, u_{r,S})$, is defined based on the game (N, v, P), where each player k with $k \neq r$ is the union P_k, and player r is the coalition S.

Using the modified game $(M, u_{r,S})$ and a value f for TU-games, the reduced game (P_r, w_r) is a TU-game with a set of players P_r and characteristic function

$$w_r(S) = f_r(M, u_{r,S}) \tag{2}$$

for any $S \subseteq P_r$.

Finally, a value g for TU-games with a priori unions is obtained by reapplying the value f over (P_r, w_r). That is

$$g_i(N, v, P) = f_i(P_r, w_r) \tag{3}$$

for all $i \in P_r$.

We call the Owen procedure to that described in equations (2) and (3) to obtain a coalitional value g for TU-games with a priori unions using a value f for TU-games.

The Owen value and the Banzhaf-Owen value are the result of applying the Owen procedure using the Shapley value (Owen 1977) and the Banzhaf value (Owen 1981), respectively.

A similar approach to obtain coalitional values in two steps is presented in Gómez-Rúa and Vidal-Puga (2010). They propose different coalitional values; in one of them, the payoffs obtained by the unions are given using a weighted Shapley value, with weights given by the union sizes.

3 The equal division value for TU-games with a priori unions

Alonso-Meijide et al. (2020) define the equal division value for TU-games with a priori unions as the natural extension of the equal division value to the set $\mathcal{G}^U$.

Definition 3.1 (Alonso-Meijide et al. 2020) The equal division value for TU-games with a priori unions ED^U is defined as

$$ED_i^U(N, v, P) = \frac{v(N)}{mp_k}$$

for all $(N, v, P) \in \mathcal{G}^U$ with $P = \{P_1, \ldots, P_m\}$ and $i \in P_k$ where p_k denotes the cardinal of P_k.

Alonso-Meijide et al. (2020) characterize the equal division value for TU-games with a priori unions using, among others, symmetry and additivity properties. In this section, we provide a second axiomatic characterization of the equal division value for TU-games with a priori unions using the concept of CED value and two additional properties: the quotient game property (Winter, 1992) and the balanced contributions in the unions' property (Vázquez-Brage et al., 1996). First, we consider these two properties.

Quotient Game Property (QGP). A value g for TU-games with a priori unions satisfies the QGP if, for all $(N, v, P) \in \mathcal{G}_N^U$ with $P = \{P_1, \ldots, P_m\}$, it holds that

$$\sum_{i \in P_k} g_i \left(N, v, P\right) = g_k \left(M, v/P, P^m\right)$$

for all $P_k \in P$, where $(M, v/P)$ is the quotient game of (N, v, P).

Balanced Contributions in the Unions (BCU). A value g for TU-games with a priori unions satisfies balanced contributions in the unions if, for all $(N, v, P) \in \mathcal{G}_N^U$ and all $i, j \in P_k$ with $P_k \in P$, it holds that

$$g_i \left(N, v, P\right) - g_i \left(N, v, P_{-j}\right) = g_j \left(N, v, P\right) - g_j \left(N, v, P_{-i}\right)$$

where P_{-l} denotes the partition $\{P_1, \ldots, P_{k-1}, P_k \setminus \{l\}, \{l\}, P_{k+1}, \ldots, P_m\}$ for all $l \in P_k$.

If a value satisfies the QGP, then the total amount received by the union players coincides with the amount obtained by the union in the game played by the unions (the quotient game). For example, the Owen value satisfies this property but the Banzhaf-Owen value does not. The balanced contributions in the unions property compare the payoff obtained by player i in the original game (N, v, P) and the game when player j of the same union decides to leave the union and stay alone (N, v, P_{-j}). This property establishes that the difference between the payoffs obtained by player i in the two previous games coincides with the same difference for player j when player i leaves the union. This property is a particular case of the splitting property (Casajus, 2009), in which this difference is the same considering any game (N, v, P'), where the partition P' is finer than P.

Vázquez-Brage et al. (1997) prove that the Owen value is the unique coalitional Shapley value that satisfies the properties of the quotient game and BCU. Similarly, Alonso-Meijide and Fiestras-Janeiro (2002) characterize the coalitional Banzhaf value as a unique coalitional Banzhaf value satisfying the properties of quotient games and BCU. In the same spirit, we present a characterization of the equal division value for TU-games with a priori unions.

The mathematical arguments of some of the proofs presented in this paper are similar to those in previous studies and they are relegated to the Appendix. They share the quotient games ideas and balanced contribution properties joint to the coalitional value concept to show the unicity of the solutions.

Theorem 3.2 *ED^U is the unique CED value that satisfies the QGP and BCU.*

In the previous theorem, the CED value can be stated as a third property. Moreover, the CED value could be replaced by any set of properties that characterizes the equal division value in the family of TU-games by adding the mention of the trivial coalition structure.

The ED^U value is an extension of ED for TU-games with a priori unions that are intuitive and natural. Moreover, let us check if it is the value obtained by the procedure to obtain coalitional values in the two steps proposed by Owen (1977) described in Subsection 2.3.

Theorem 3.3 *The equal division value with a priori unions ED^U is the result of applying the Owen procedure using the equal division value ED.*

4 Three equal surplus division values for TU-games with a priori unions

Alonso-Meijide et al. (2020) proposed three alternative ways to extend the equal surplus division value to TU-games with a priori unions. In this section, we provide new characterizations of these coalitional values.

4.1 The equal surplus division value 1

The equal surplus division value 1 divides the grand coalition value in the quotient game using the equal surplus division value and then equally divides the amount assigned to each union among its members.

Definition 4.1 (Alonso-Meijide et al. 2020) The equal surplus division value (one) for TU-games with a priori unions $ESD1^U$ is defined by

$$ESD1_i^U(N,v,P) = \frac{(v/P)(k)}{p_k} + \frac{(v/P)^0(M)}{mp_k} = \frac{v(P_k)}{p_k} + \frac{v(N) - \sum_{l\in M} v(P_l)}{mp_k}$$

for all $(N,v,P) \in \mathcal{G}^U$ with $P = \{P_1, \ldots, P_m\}$ and $i \in P_k$.

Notice that it can be easily checked that $ESD1^U$ is a *coalitional equal surplus division value (CESD)*, as

$$ESD1^U(N,v,P^n) = ESD(N,v)$$

for all $(N,v) \in \mathcal{G}$.

We use this feature to provide a new characterization of $ESD1^U$ in the remainder of this subsection. First, we define a new property.

Equality Inside Unions (EIU). Value g for TU-games with a priori unions satisfies EIU if, for all $(N,v,P) \in \mathcal{G}^U$, all $P_k \in P$, and $i,j \in P_k$, it holds that $g_i(N,v,P) - g_j(N,v,P) = 0$.

The previous property takes up the egalitarianism idea inside unions. This property considers that players within a union are willing to show strict equality. It can be seen as a stronger symmetry property version, as all players belonging to the same union obtain the same payoff without considering the characteristic function of the game.

Theorem 4.2 *$ESD1^U$ is the unique CESD value for TU-games with a priori unions satisfying QGP and EIU.*

It can be easily proved that the equal division value for TU-games with a priori unions satisfies EIU. Moreover, EIU could replace BCU in Theorem 3.2 to obtain a new characterization of the equal division value for TU-games with a priori unions.

4.2 The equal surplus division value 2

The equal surplus division value 2 again divides the value of the grand coalition in the quotient game using the equal surplus division value; then, it distributes the amount $v(P_k)$ assigned to each union P_k giving $v(i)$ to each player $i \in P_k$ and dividing $v(P_k) - \sum_{j \in P_k} v(j)$ equally among the players in P_k.

Definition 4.3 (Alonso-Meijide et al. 2020) The equal surplus division value (two) for TU-games with a priori unions $ESD2^U$ is defined by

$$ESD2_i^U(N, v, P) = v(i) + \frac{v(P_k) - \sum_{j \in P_k} v(j)}{p_k} + \frac{v(N) - \sum_{l \in M} v(P_l)}{m p_k}$$

for all $(N, v, P) \in \mathcal{G}^U$, with $P = \{P_1, \ldots, P_m\}$ and $i \in P_k$.

Notice that it can be easily checked that $ESD2^U$ is a $CESD$, as

$$ESD2^U(N, v, P^n) = ESD(N, v)$$

for all $(N, v) \in \mathcal{G}$.

We use this feature to provide a new $ESD2^U$ characterization in the remainder of this subsection. First, we define a new property.

Difference Maintenance of Individual Values Inside Unions (DMIVIU). A value g for TU-games with a priori unions satisfies DMIVIU if, for all $(N, v, P) \in \mathcal{G}^U$, all $P_k \in P$, and $i, j \in P_k$, it holds that $g_i(N, v, P) - g_j(N, v, P) = v(i) - v(j)$.

This property is similar to EIU, as DMIVIU is a stronger version of the symmetry property, but it is not as strong as EIU. In this case, the difference between the payoffs of two players of the same union coincides with the difference between the amounts given by the characteristic function of the game to individual coalitions, without considering the amounts given to coalitions with two or more players. It is evident that in the case of zero-normalized games, DMIVIU is equivalent to EIU.

Theorem 4.4 *$ESD2^U$ is the unique CESD value for TU-games with a priori unions satisfying QGP and DMIVIU.*

4.3 The equal surplus division value 3

Finally, the equal surplus division value 3 assigns $v(i)$ to each player i and then divides $v^0(N)$ among the players using ED^U.

Definition 4.5 (Alonso-Meijide et al. 2020) The equal surplus division value (three) for TU-games with a priori unions $ESD3^U$ is defined by

$$ESD3_i^U(N,v,P) = v(i) + ED^U(N,v^0,P) = v(i) + \frac{v(N) - \sum_{j\in N} v(j)}{mp_k}$$

for all $(N,v,P) \in \mathcal{G}^U$ with $P = \{P_1,\ldots,P_m\}$ and $i \in P_k$.

Notice that it can be easily checked that $ESD3^U$ is a *CESD*, as

$$ESD3^U(N,v,P^n) = ESD(N,v)$$

for all $(N,v) \in \mathcal{G}$.

We use this feature to provide a new characterization of $ESD3^U$ in the remainder of this section. Nevertheless, $ESD3^U$ does not satisfy QGP; therefore, we introduce a newly modified quotient game.

Let $(N,v,P) \in \mathcal{G}^U$ with $P = \{P_1,...,P_m\}$ and denote $M = \{1,...,m\}$. The quotient* game of (N,v,P) is the TU-game $(M,\bar{v}/P)$, where

$$(\bar{v}/P)(R) = \begin{cases} \sum_{k\in R}\sum_{i\in P_k} v(i) & \text{if } R \subset M \\ v(N) & \text{if } R = M \end{cases}$$

Quotient* Game Property (Q*GP). A value g for TU-games with a priori unions satisfies the Q*GP if for all $(N,v,P) \in \mathcal{G}^U$ with $P = \{P_1,...,P_m\}$, it holds that

$$\sum_{i\in P_k} g_i(N,v,P) = g_k(M,\bar{v}/P,P^m)$$

for all $P_k \in P$, where $(M,\bar{v}/P)$ is the quotient* game of (N,v,P).

Theorem 4.6 *$ESD3^U$ is the unique CESD value for TU-games with a priori unions satisfying Q*GP and BCU.*

5 Two new extensions of the equal surplus division value

In this section, we introduce two new extensions of the equal surplus division value for TU-games with a priori unions. First is the value obtained by applying the Owen procedure using the equal surplus division value. The other is an extension of the equal surplus division value that satisfies the quotient game and BCU properties.

5.1 Coalitional value using equal surplus division value in two steps

The first new extension is obtained by applying the procedure in two steps using an equal surplus division value. The first part of the value coincides with $ESD2^U$, as mentioned later. In the second part, it allocates the difference between the average value of the players in the union and the value of the player, then the difference between the value of the player's contribution to the grand coalition minus the union and the average contribution of the players in the union to the grand coalition minus the union, all divided by the unions total.

Definition 5.1 The equal surplus division value (four) for TU-games with a priori unions $ESD4^U$ is defined by

$$ESD4_i^U\left(N,v,P\right) = v(i) + \frac{v(P_k) - \sum_{j \in P_k} v(j)}{p_k} + \frac{v(N) - \sum_{l \in M} v(P_l)}{mp_k} +$$
$$\frac{1}{m}\left(\sum_{t \in P_k} \frac{v\left(t\right)}{p_k} - v\left(i\right)\right) + \frac{1}{m}\left(v\left(\cup_{r \in M \setminus k} P_r \cup i\right) - \sum_{t \in P_k} \frac{v\left(\cup_{r \in M \setminus k} P_r \cup t\right)}{p_k}\right)$$

for all $(N,v,P) \in \mathcal{G}^U$ with $P = \{P_1, \ldots, P_m\}$ and $i \in P_k$.

Using the procedure proposed by Owen (1977) shown in Section 2.3, where ESD is now used in games (2) and (3), we can obtain the solution $g = ESD4^U$. Let us consider this in the following theorem.

Theorem 5.2 *The equal surplus division value with a priori unions $ESD4^U$ is the result of applying the Owen procedure using the equal surplus division value ESD.*

Remark 5.3 Note that $ESD4$ can be written in terms of $ESD2$

$$ESD4_i^U(N,v,P) = ESD2_i^U(N,v,P) + \frac{1}{m}\left(\sum_{t\in P_r}\frac{v(t)}{p_r} - v(i)\right) +$$
$$\frac{1}{m}\left(v\left(\cup_{k\in M\backslash r}P_k \cup i\right) - \sum_{t\in P_r}\frac{v\left(\cup_{k\in M\backslash r}P_k \cup t\right)}{p_r}\right)$$

for all $(N,v,P)\in\mathcal{G}^U$ with $P=\{P_1,\ldots,P_m\}$ and $i\in P_k$.

It can be easily checked that

$$ESD4_i^U(N,v,P^n) = ESD_i(N,v)$$

and

$$\sum_{i\in P_r} ESD4_i^U(N,v,P) = ESD_r(M,v/P),$$

that is, $ESD4^U$ is a CESD value that satisfies the QGP.

The value $ESD4$ is a CESD value that satisfies the quotient game. To characterize $ESD4^U$ in the proposed context, we define the following property:

Balanced contributions because of the players abandonment in the union (BCPA). A value g for TU-games with a priori unions satisfies BCPA if, for all $(N,v,P)\in\mathcal{G}^U$ and all $i,j\in P_k$ with $P_k\in P$, it holds that

$$g_i(N,v,P) - g_i\left(N\backslash P_k\cup i, v_{N\backslash P_k\cup i}, P\backslash P_k\cup\{i\}\right) =$$
$$g_j(N,v,P) - g_j\left(N\backslash P_k\cup j, v_{N\backslash P_k\cup j}, P\backslash P_k\cup\{j\}\right)$$

where the game $\left(N\backslash P_k\cup i, v_{N\backslash P_k\cup i}, P\backslash P_k\cup\{i\}\right)$ is defined as $v_{N\backslash P_k\cup i}(S) = v(S)$ for all $S\subseteq N\backslash P_k\cup i$.

This new property states that given two players in the same union, they get the same difference between the payoff of the original game and the payoff of the game where all the players of the union leave. This property has a similar interpretation to the balanced contributions property.

Theorem 5.4 *$ESD4^U$ is the unique CESD value for TU-games with a priori unions satisfying QGP and BCPA.*

5.2 Coalitional equal surplus division value satisfying balanced contributions and quotient game

In this section, we define a value for TU-games with a priori unions that extends the equal surplus division value and satisfies the QGP and BCU. The first part of the value coincides with the $ESD1^U$, as mentioned later, and a weighted difference between the value of the subsets of the union that contain the player minus the subsets that do not.

Definition 5.5 The equal surplus division value (five) for TU-games with a priori unions $ESD5^U$ is defined by

$$ESD5_i^U(N,v,P) = \frac{v(P_k)}{p_k} + \frac{v(N) - \sum_{l\in M} v(P_l)}{mp_k} + \sum_{\substack{T\subset P_k \\ i\in T}} \frac{P^{m,p_k,t}}{t} v(T) - \sum_{\substack{T\subset P_k \\ i\notin T}} \frac{P^{m,p_k,t}}{p_k - t} v(T)$$

for all $(N,v,P) \in \mathcal{G}^U$ with $P = \{P_1, \ldots, P_m\}$ and $i \in P_k$; where $t = |T|$ and

$$P^{m,p_k,t} = \frac{1}{2} \qquad \text{if } p_k = 2 \text{ and } t = 1,$$

$$P^{m,p_k,t} = \frac{1}{p_k}\left(1 + \sum_{j=1}^{p_k-2} \frac{1}{m+j}\right) \qquad \text{if } p_k > 2 \text{ and } t = 1,$$

$$P^{m,p_k,t} = \frac{m}{(m+1)p_k} \qquad \text{if } p_k > 2 \text{ and } t = p_k - 1,$$

$$P^{m,p_k,t} = \frac{m+(z-1)}{(p_k-(z-1))(m+z)}\left(\sum_{j=0}^{z-2} \frac{p_k - j - t}{p_k - j}\right)$$

if $p_k > 3$ and $t = (p_k - z)$ such that $z \in \{2, ..., p_k - 2\}$.

Remark 5.6 Note that $ESD5$ can be written in terms of $ESD1$

$$ESD5_i^U(N,v,P) = ESD1_i^U(N,v,P) + \sum_{\substack{T\subset P_k \\ i\in T}} \frac{P^{m,p_k,t}}{t} v(T) - \sum_{\substack{T\subset P_k \\ i\notin T}} \frac{P^{m,p_k,t}}{p_k - t} v(T).$$

for all $(N,v,P) \in \mathcal{G}^U$ with $P = \{P_1, \ldots, P_m\}$ and $i \in P_k$.

In the last result of this study, we characterized the $ESD5$ value in the same spirit as an axiomatic characterization of the Owen value given in Vázquez-Brage et al. (1997).

Theorem 5.7 *$ESD5^U$ is the unique CESD value that satisfies the QGP and BCU.*

Table 1 shows the summary of the properties fulfilled by all the values. Note that all theorems are based on independent properties. The logical demonstration of the independence of the axioms is provided as an online supplement to readers wishing to request them.

	CED	$CESD$	QGP	Q^*GP	BCU	EIU	$DMIVIU$	$BCPA$
ED^U	✓	–	✓	–	✓	–	–	–
$ESD1^U$	–	✓	✓	–	–	✓	–	–
$ESD2^U$	–	✓	✓	–	–	–	✓	–
$ESD3^U$	–	✓	–	✓	✓	–	–	–
$ESD4^U$	–	✓	✓	–	–	–	–	✓
$ESD5^U$	–	✓	✓	–	✓	–	–	–

Table 1: Properties satisfied by all the values

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Appendix

Here, the reader can find the proofs of the theorems stated in this study.

Proof of Theorem 3.2.

Proof. Consider a TU-game with a priori unions $(N, v, P) \in \mathcal{G}^U$ such that $P = \{P_1, ..., P_m\}$ and denote $M = \{1, ..., m\}$. We check that ED^U satisfies the QGP. For all $k \in M$, we have

$$\sum_{i \in P_k} ED_i^U(N, v, P) = \sum_{i \in P_k} \frac{v(N)}{mp_k} = \frac{v(N)}{m}$$

and

$$ED_k^U(M, v/P, P^m) = \frac{(v/P)(M)}{m} = \frac{v(N)}{m}.$$

Let us check that ED^U satisfies BCU. For all $i, j \in P_k$, we have that

$$ED_i^U(N, v, P) - ED_i^U(N, v, P_{-j}) = \frac{v(N)}{mp_k} - \frac{v(N)}{(m+1)(p_k - 1)}$$

and

$$ED_j^U(N, v, P) - ED_j^U(N, v, P_{-i}) = \frac{v(N)}{mp_k} - \frac{v(N)}{(m+1)(p_k - 1)}.$$

Finally, the uniqueness is proven analogously as the uniqueness in Theorem 2 of Vázquez-Brage et al. (1997). Let us suppose that there exist two different CED values, f^1 and f^2, satisfying the QGP and BCU. We can find a coalitional game (N, v, P), where P is the maximal number of unions, such that $f^1(N, v, P) \neq f^2(N, v, P)$. Considering that f^1 and f^2 satisfy the QGP, $\forall P_k \in P$ and $l \in \{1, 2\}$, we have

$$\sum_{i \in P_k} f_i^l(N, v, P) = f_k^l(M, v/P, P^m).$$

But f^1 and f^2 are CED values, then

$$\sum_{i\in P_k} f_i^1(N,v,P) = \sum_{i\in P_k} f_i^2(N,v,P) = ED_k(M,v/P). \tag{4}$$

If P_k is such that $|P_k| = 1$, i.e. $P_k = \{i\}$, then

$$f_i^1(N,v,P) = f_i^2(N,v,P)$$

However, if $|P_k| > 1$, for any $i,j \in P_k$, we have by BCU

$$f_i^l(N,v,P) - f_j^l(N,v,P) = f_i^l(N,v,P_{-j}) - f_j^l(N,v,P_{-i})$$

for all $l \in \{1,2\}$. Therefore, the maximality of P implies that

$$f_i^1(N,v,P) - f_j^1(N,v,P) = f_i^2(N,v,P) - f_j^2(N,v,P)$$

and we have

$$f_i^1(N,v,P) - f_i^2(N,v,P) = A^k$$

for all $i \in P_k$. By equation 4, we have $A^k = 0$ and

$$f_i^1(N,v,P) = f_i^2(N,v,P)$$

Thus, $f^1(N,v,P) = f^2(N,v,P)$, and we have proven this uniqueness. □

Proof of Theorem 3.3.

Proof. Consider a TU-game with a priori unions $(N,v,P) \in \mathcal{G}^U$ such that $P = \{P_1,...,P_m\}$ and denote $M = \{1,...,m\}$. Given a coalition $S \subseteq P_r$, we can obtain the reduced game (2) by applying ED to the modified game (1),

$$w_r(S) = ED_r(M,u_{r,S}) = \frac{u_{r,S}(.M)}{m} = \frac{v\left(\cup_{k\in H\setminus r} P_k \cup S\right)}{m}.$$

Again, if we reapply ED to the reduced game (2) as (3), for all players $i \in P_r$, we obtain

$$ED_i(P_r,w_r) = \frac{w_r(P_r)}{p_r} = \frac{v\left(\cup_{k\in H\setminus r} P_k \cup P_r\right)/m}{p_r} = \frac{v(N)}{mp_r} = ED_i^U(N,v,P).$$

□

Proof of Theorem 4.2.

Proof. Consider a TU-game with a priori unions $(N, v, P) \in \mathcal{G}^U$ such that $P = \{P_1, ..., P_m\}$ and denote $M = \{1, ..., m\}$. Let us check that $ESD1^U$ satisfies the QGP. For all $k \in M$, we have

$$\begin{aligned}\sum_{i \in P_k} ESD1_i^U(N, v, P) &= \sum_{i \in P_k} \left(\frac{v(P_k)}{p_k} + \frac{v(N) - \sum_{l \in M} v(P_l)}{mp_k} \right) \\ &= v(P_k) + \frac{v(N) - \sum_{l \in M} v(P_l)}{m}\end{aligned}$$

and

$$\begin{aligned}ESD1_k^U(M, v/P, P^m) &= \frac{(v/P)(k)}{1} + \frac{(v/P)(M) - \sum_{l \in M} (v/P)(l)}{m} \\ &= v(P_k) + \frac{v(N) - \sum_{l \in M} v(P_l)}{m}.\end{aligned}$$

Let us check that $ESD1^U$ satisfies the EIU. For all $i, j \in P_k$, we have

$$ESD1_i^U(N, v, P) - ESD1_j^U(N, v, P) = 0.$$

Finally, the uniqueness is proven analogously as the uniqueness in Theorem 2 of Vázquez-Brage et al. (1997). □

Proof of Theorem 4.4.

Proof. Consider a TU-game with a priori unions $(N, v, P) \in \mathcal{G}^U$ such that $P = \{P_1, ..., P_m\}$ and denote $M = \{1, ..., m\}$. Let us check that $ESD2^U$ satisfies the QGP. For all $k \in M$, we have

$$\begin{aligned}\sum_{i \in P_k} ESD2_i^U(N, v, P) &= \sum_{i \in P_k} \left(v(i) + \frac{v(P_k) - \sum_{j \in P_k} v(j)}{p_k} + \frac{v(N) - \sum_{l \in M} v(P_l)}{mp_k} \right) \\ &= \sum_{i \in P_k} v(i) + v(P_k) - \sum_{j \in P_k} v(j) + \frac{v(N) - \sum_{l \in M} v(P_l)}{m} \\ &= v(P_k) + \frac{v(N) - \sum_{l \in M} v(P_l)}{m}\end{aligned}$$

and

$$ESD2^U_k\,(M, v/P, P^m) = (v/P)(k) + \frac{(v/P)(k) - (v/P)(k)}{1} + \frac{(v/P)(M) - \sum_{l\in M}(v/P)(l)}{m}$$
$$= v(P_k) + \frac{v(N) - \sum_{l\in M} v(P_l)}{m}.$$

Let us check that $ESD2^U$ satisfies DMIVIU. For all $i, j \in P_k$, we have

$$ESD2^U_i\,(N, v, P) - ESD2^U_j\,(N, v, P) = v(i) - v(j).$$

Finally, the uniqueness is proven analogously as the uniqueness in Theorem 2 of Vázquez-Brage et al. (1997). □

Proof of Theorem 4.6.

Proof. Consider a TU-game with a priori unions $(N, v, P) \in \mathcal{G}^U$ such that $P = \{P_1, ..., P_m\}$ and denote $M = \{1, ..., m\}$. Let us check that $ESD3^U$ satisfies Q*GP. For all $k \in M$, we have

$$\sum_{i\in P_k} ESD3^U_i\,(N, v, P) = \sum_{i\in P_k}\left(v(i) + \frac{v(N) - \sum_{j\in N} v(j)}{mp_k}\right)$$
$$= \sum_{i\in P_k} v(i) + \frac{v(N) - \sum_{j\in N} v(j)}{m}$$

and

$$ESD3^U_k\,(M, \bar{v}/P, P^m) = (\bar{v}/P)(k) + \frac{(\bar{v}/P)(M) - \sum_{l\in M}(\bar{v}/P)(l)}{m}$$
$$= \sum_{i\in P_k} v(i) + \frac{v(N) - \sum_{k\in M}\sum_{j\in P_k} v(j)}{m}$$
$$= \sum_{i\in P_k} v(i) + \frac{v(N) - \sum_{j\in N} v(j)}{m}.$$

Let us check that $ESD3^U$ satisfies BCU. For all $i, j \in P_k$, we have

$$ESD3^U_i\,(N, v, P) - ESD3^U_i\,(N, v, P_{-j}) = \frac{v(N) - \sum_{j\in N} v(j)}{mp_k} - \frac{v(N) - \sum_{j\in N} v(j)}{(m+1)(p_k-1)}$$

and

$$ESD3_j^U(N,v,P) - ESD3_j^U(N,v,P_{-i}) = \frac{v(N) - \sum_{j\in N} v(j)}{mp_k} - \frac{v(N) - \sum_{j\in N} v(j)}{(m+1)(p_k-1)}.$$

Finally, the uniqueness is proven in a similar way as the uniqueness in Theorem 2 of Vázquez-Brage et al. (1997). □

Proof of Theorem 5.2.

Proof. Consider a TU-game with a priori unions $(N,v,P) \in \mathcal{G}^U$ such that $P = \{P_1, ..., P_m\}$ and denote $M = \{1, ..., m\}$. First, ESD is applied to the modified game (1) to obtain the reduced game (2), where only the individual payoffs $u_{r,S}(i)$ and the total payoff $u_{r,S}(M)$ are necessary. Therefore, for all unions P_r

$$w_r(S) = ESD_r(M, u_{r,S}) = u_{r,S}(r) + \frac{u_{r,S}(M) - \sum_{l\in M} u_{r,S}(l)}{m} =$$
$$v(S) + \frac{v\left(\cup_{k\in M\setminus r} P_k \cup S\right) - \sum_{l\in M\setminus r} v(P_l) - v(S)}{m}.$$

Taking again ESD and applying it to the reduced game (2) as (3), for all players $i \in P_r$,

$$ESD_i(P_r, w_r) = w_r(i) + \frac{w_r(P_r) - \sum_{t\in P_r} w_r(t)}{p_r} =$$
$$v(i) + \frac{v\left(\cup_{k\in M\setminus r} P_k \cup i\right) - \sum_{l\in M\setminus r} v(P_l) - v(i)}{m} +$$
$$\frac{v(P_r) + \frac{v\left(\cup_{k\in M\setminus r} P_k \cup P_r\right) - \sum_{l\in M\setminus r} v(P_l) - v(P_r)}{m} - \sum_{t\in P_r}\left(v(t) + \frac{v\left(\cup_{k\in M\setminus r} P_k \cup t\right) - \sum_{l\in M\setminus r} v(P_l) - v(t)}{m}\right)}{p_r}.$$

As

$$v\left(\cup_{k\in M\setminus r} P_k \cup P_r\right) - \sum_{l\in M\setminus r} v(P_l) - v(P_r) = v(N) - \sum_{l\in M} v(P_l)$$

we have

$$ESD_i(P_r, w_r) = v(i) + \frac{v\left(\cup_{k\in M\setminus r} P_k \cup i\right)}{m} - \frac{v(i)}{m} - \frac{\sum_{l\in M\setminus r} v(P_l)}{m} + \frac{v(P_r)}{p_r} +$$
$$\frac{v(N)}{mp_r} - \frac{\sum_{l\in M} v(P_l)}{mp_r} - \sum_{t\in P_r} \frac{v(t)}{p_r} - \sum_{t\in P_r} \frac{v\left(\cup_{k\in M\setminus r} P_k \cup t\right)}{mp_r} + \sum_{t\in P_r}\sum_{l\in M\setminus r} \frac{v(P_l)}{mp_r} + \sum_{t\in P_r} \frac{v(t)}{mp_r}.$$

The second last term can be written as

$$\sum_{t\in P_r}\sum_{l\in M\setminus r} \frac{v(P_l)}{mp_r} = \sum_{l\in M\setminus r} p_r \frac{v(P_l)}{mp_r} = \sum_{l\in M\setminus r} \frac{v(P_l)}{m}$$

and then, we obtain

$$ESD_i(P_r, w_r) = v(i) + \frac{v\left(\cup_{k\in M\setminus r} P_k \cup i\right).}{m} - \frac{v(i)}{m} + \frac{v(P_r)}{p_r} +$$
$$\frac{v(N)}{mp_r} - \frac{\sum_{l\in M} v(P_l)}{mp_r} - \sum_{t\in P_r} \frac{v(t)}{p_r} - \sum_{t\in P_r} \frac{v\left(\cup_{k\in M\setminus r} P_k \cup t\right)}{mp_r} + \sum_{t\in P_r} \frac{v(t)}{mp_r}.$$

Reordering terms, we have

$$ESD_i(P_r, w_r) = v(i) + \frac{1}{m}\left(\sum_{t\in P_r} \frac{v(t).}{p_r} - v(i)\right) + \frac{1}{p_r}\frac{v(N) - \sum_{l\in M} v(P_l)}{m} +$$
$$\frac{1}{p_r}\left(v(P_r) - \sum_{t\in P_r} v(t)\right) + \frac{1}{m}\left(v\left(\cup_{k\in M\setminus r} P_k \cup i\right). - \sum_{t\in P_r} \frac{v\left(\cup_{k\in M\setminus r} P_k \cup t\right)}{p_r}\right)$$
$$= ESD4_i^U(N, v, P).$$

□

Proof of Theorem 5.4.

Proof. Consider a TU-game with a priori unions $(N, v, P) \in \mathcal{G}^U$ such that $P = \{P_1, ..., P_m\}$ and denote $M = \{1, ..., m\}$. Let us check that $ESD4^U$ satisfies the BCPA. For all $l \in M$ and all $i, j \in P_l$,

$$ESD4_i^U(N, v, P) - ESD4_j^U(N, v, P) = ESD2_i^U(N, v, P) + \frac{1}{m}\left(\sum_{t\in P_l}\frac{v(t).}{p_l} - v(i)\right) +$$
$$\frac{1}{m}\left(v\left(\cup_{k\in M\setminus l}P_k \cup i\right) - \sum_{t\in P_l}\frac{v\left(\cup_{k\in M\setminus l}P_k \cup t\right)}{p_l}\right) - ESD2_j^U(N, v, P) -$$
$$\frac{1}{m}\left(\sum_{t\in P_l}\frac{v(t)}{p_l} - v(j)\right) - \frac{1}{m}\left(v\left(\cup_{k\in M\setminus l}P_k \cup j\right) - \sum_{t\in P_l}\frac{v\left(\cup_{k\in M\setminus l}P_k \cup t\right)}{p_l}\right).$$

By the DMIVIU property that satisfies $ESD2^U$, we have $ESD2_i - ESD2_j = v(i) - v(j)$. Then we have

$$ESD4_i^U(N, v, P) - ESD4_j^U(N, v, P) =$$
$$v(i) - v(j) - \frac{v(i)}{m} + \frac{v(j)}{m} + \frac{v\left(\cup_{k\in M\setminus l}P_k \cup i\right)}{m} - \frac{v\left(\cup_{k\in M\setminus l}P_k \cup j\right)}{m}.$$

However,

$$ESD4_i^U\left(N\setminus P_l \cup i, v_{N\setminus P_l\cup i}, P\setminus P_l \cup \{i\}\right) - ESD4_j^U\left(N\setminus P_l \cup j, v_{N\setminus P_l\cup j}, P\setminus P_l \cup \{j\}\right) =$$
$$ESD2_i^U\left(N\setminus P_l \cup i, v_{N\setminus P_l\cup i}, P\setminus P_l \cup \{i\}\right) + \frac{1}{m}\left(\frac{v_{N\setminus P_l\cup i}(i)}{1} - v_{N\setminus P_l\cup i}(i)\right) +$$
$$\frac{1}{m}\left(v_{N\setminus P_l\cup i}\left(\cup_{P_k\in P\setminus P_l}P_k \cup i\right) - \frac{v_{N\setminus P_l\cup i}\left(\cup_{P_k\in P\setminus P_l}P_k \cup i\right)}{1}\right) -$$
$$ESD2_j^U\left(N\setminus P_l \cup j, v_{N\setminus P_l\cup j}, P\setminus P_l \cup \{j\}\right) - \frac{1}{m}\left(\frac{v_{N\setminus P_l\cup j}(j)}{1} - v_{N\setminus P_l\cup j}(j)\right) -$$
$$\frac{1}{m}\left(v_{N\setminus P_l\cup j}\left(\cup_{P_k\in P\setminus P_l}P_k \cup j\right) - \frac{v_{N\setminus P_k\cup j}\left(\cup_{P_k\in P\setminus P_l}P_k \cup j\right)}{1}\right).$$

We have

$$ESD4_i^U\left(N\setminus P_l \cup i, v_{N\setminus P_l\cup i}, P\setminus P_l \cup \{i\}\right) - ESD4_j^U\left(N\setminus P_l \cup j, v_{N\setminus P_l\cup j}, P\setminus P_l \cup \{j\}\right) =$$
$$ESD2_i^U\left(N\setminus P_l \cup i, v_{N\setminus P_l\cup i}, P\setminus P_l \cup \{i\}\right) - ESD2_j^U\left(N\setminus P_l \cup j, v_{N\setminus P_l\cup j}, P\setminus P_l \cup \{j\}\right)$$

and then

$$ESD2_i^U\left(N\backslash P_l \cup i, v_{N\backslash P_l\cup i}, P\backslash P_l \cup \{i\}\right) - ESD2_j^U\left(N\backslash P_l \cup j, v_{N\backslash P_l\cup j}, P\backslash P_l \cup \{j\}\right) =$$
$$v_{N\backslash P_l\cup i}(i) + \frac{v_{N\backslash P_l\cup i}(i) - v_{N\backslash P_l\cup i}(i)}{1} + \frac{v_{N\backslash P_l\cup i}(N\backslash P_l \cup i) - (\sum_{P_k\in P\backslash P_l\cup i} v_{N\backslash P_l\cup i}(P_k))}{m} -$$
$$v_{N\backslash P_l\cup j}(j) - \frac{v_{N\backslash P_l\cup j}(j) - v_{N\backslash P_l\cup j}(j)}{1} - \frac{v_{N\backslash P_l\cup j}(N\backslash P_l \cup j) - (\sum_{P_k\in P\backslash P_l\cup j} v_{N\backslash P_l\cup j}(P_k))}{m} =$$
$$v_{N\backslash P_l\cup i}(i) - v_{N\backslash P_l\cup j}(j) + \frac{v_{N\backslash P_l\cup i}(N\backslash P_l \cup i)}{m} - \frac{v_{N\backslash P_l\cup j}(N\backslash P_l \cup j)}{m} - \frac{\sum_{P_k\in P\backslash P_l\cup i} v_{N\backslash P_l\cup i}(P_k)}{m} +$$
$$\frac{\sum_{P_k\in P\backslash P_l\cup j} v_{N\backslash P_l\cup j}(P_k)}{m} = v(i) - v(j) + \frac{v\left(\cup_{k\in M\backslash l} P_k \cup i\right)}{m} - \frac{v\left(\cup_{k\in M\backslash l} P_k \cup j\right)}{m} - \frac{v(i)}{m} + \frac{v(j)}{m}.$$

Finally, the uniqueness is proven similarly as the uniqueness in Theorem 2 of Vázquez-Brage et al. (1997). □

Proof of Theorem 5.7.

Proof. Consider a TU-game with a priori unions $(N, v, P) \in \mathcal{G}^U$ such that $P = \{P_1, ..., P_m\}$ and denote $M = \{1, ..., m\}$. Let us check that $ESD5^U$ satisfies the QGP. For all $k \in M$, we have

$$\sum_{i\in P_k} ESD5_i^U(N, v, P) =$$
$$\sum_{i\in P_k}\left(\frac{v(P_k)}{p_k} + \frac{v(N) - \sum_{l\in M} v(P_l)}{mp_k}\right) + \sum_{i\in P_k}\sum_{\substack{T\subset P_k\\ i\in T}} \frac{P^{m,p_k,t}}{t} v(T) - \sum_{i\in P_k}\sum_{\substack{T\subset P_k\\ i\notin T}} \frac{P^{m,p_k,t}}{p_k - t} v(T) =$$
$$v(P_k) + \frac{v(N) - \sum_{l\in M} v(P_l)}{m} + t\sum_{T\subset P_k} \frac{P^{m,p_k,t}}{t} v(T) - (p_k - t)\sum_{T\subset P_k} \frac{P^{m,p_k,t}}{p_k - t} v(T) =$$
$$v(P_k) + \frac{v(N) - \sum_{l\in M} v(P_l)}{m}.$$

It is immediate that

$$ESD5_k^U(M, v/P, P^m) = v(P_k) + \frac{v(N) - \sum_{l\in M} v(P_l)}{m}.$$

Let us check that $ESD5^U$ satisfies BCU. For all $k \in M$ and all $i, j \in P_k$,

$$ESD5_i^U(N,v,P) - ESD5_j^U(N,v,P) =$$

$$\sum_{\substack{T\subset P_k\\ i\in T}} \frac{P^{m,p_k,t}}{t} v(T) - \sum_{\substack{T\subset P_k\\ i\notin T}} \frac{P^{m,p_k,t}}{p_k - t} v(T) - \sum_{\substack{T\subset P_k\\ j\in T}} \frac{P^{m,p_k,t}}{t} v(T) + \sum_{\substack{T\subset P_k\\ j\notin T}} \frac{P^{m,p_k,t}}{p_k - t} v(T), \qquad \sum_{\substack{T\subseteq P_k\backslash j\\ i\in T}} \frac{P^{m,p_k,t}}{t}$$

$$p_k \sum_{\substack{T\subseteq P_k\backslash j\\ i\in T}} \frac{P^{m,p_k,t}}{(p_k - t)t} v(T) - \sum_{\substack{T\subseteq P_k\backslash i\\ j\in T}} \frac{P^{m,p_k,t}}{(p_k - t)t} v(T).$$

We have

$$ESD5_i^U(N,v,P_{-j}) - ESD5_j^U(N,v,P_{-i}) =$$

$$\frac{v(P_k\backslash j)}{p_k - 1} + \frac{v(N) - \sum_{P_l\in P_{-j}} v(P_l)}{(m+1)(p_k-1)} - \frac{v(P_k\backslash i)}{p_k - 1} - \frac{v(N) - \sum_{P_l\in P_{-i}} v(P_l)}{(m+1)(p_k-1)} + \sum_{\substack{T\subset P_k\backslash j\\ i\in T}} \frac{P^{m+1,p_k-1,t}}{t} v(T)$$

$$- \sum_{\substack{T\subset P_k\backslash j\\ i\notin T}} \frac{P^{m+1,p_k-1,t}}{p_k - t} v(T) - \sum_{\substack{T\subset P_k\backslash i\\ j\in T}} \frac{P^{m+1,p_k-1,t}}{t} v(T) + \sum_{\substack{T\subset P_k\backslash i\\ j\notin T}} \frac{P^{m+1,p_k-1,t}}{p_k - t} v(T) =$$

$$\frac{v(P_k\backslash j)}{p_k - 1} - \frac{v(P_k\backslash i)}{p_k - 1} - \frac{v(P_k\backslash j) + v(j)}{(m+1)(p_k-1)} + \frac{v(P_k\backslash i) + v(i)}{(m+1)(p_k-1)} + \sum_{\substack{T\subset P_k\backslash j\\ i\in T}} \frac{P^{m+1,p_k-1,t}}{t} v(T)$$

$$- \sum_{\substack{T\subset P_k\backslash j\\ i\notin T}} \frac{P^{m+1,p_k-1,t}}{p_k - t} v(T) - \sum_{\substack{T\subset P_k\backslash i\\ j\in T}} \frac{P^{m+1,p_k-1,t}}{t} v(T) + \sum_{\substack{T\subset P_k\backslash i\\ j\notin T}} \frac{P^{m+1,p_k-1,t}}{p_k - t} v(T) =$$

$$\frac{m\cdot v(P_k\backslash j)}{(m+1)(p_k-1)} - \frac{m\cdot v(P_k\backslash i)}{(m+1)(p_k-1)} + \frac{v(i)}{(m+1)(p_k-1)} - \frac{v(j)}{(m+1)(p_k-1)} + \sum_{\substack{T\subset P_k\backslash j\\ i\in T}} \frac{P^{m+1,p_k-1,t}}{t} v(T)$$

$$- \sum_{\substack{T\subset P_k\backslash j\\ i\notin T}} \frac{P^{m+1,p_k-1,t}}{p_k - t} v(T) - \sum_{\substack{T\subset P_k\backslash i\\ j\in T}} \frac{P^{m+1,p_k-1,t}}{t} v(T) + \sum_{\substack{T\subset P_k\backslash i\\ j\notin T}} \frac{P^{m+1,p_k-1,t}}{p_k - t} v(T) =$$

$$\frac{m\cdot v(P_k\backslash j)}{(m+1)(p_k-1)} - \frac{m\cdot v(P_k\backslash i)}{(m+1)(p_k-1)} + \frac{v(i)}{(m+1)(p_k-1)} - \frac{v(j)}{(m+1)(p_k-1)} +$$

$$\sum_{\substack{T\subset P_k\backslash j\\ i\in T}} \frac{P^{m+1,p_k-1,t}}{t} v(T) - \sum_{\substack{T\subset P_k\backslash i\\ j\in T}} \frac{P^{m+1,p_k-1,t}}{t} v(T).$$

Let us see that the two equations are identical. We only need to see that for player i and any coalition $T \subset P_k \backslash j$ such that $i \in T$, the weights coincide. Consider all the different cases. Note that as $i, j \in P_k$, then $p_k \geq 2$.

Case i) $p_k = 2$ then

$$p_k \sum_{\substack{T \subseteq P_k \backslash j \\ i \in T}} \frac{P^{m,p_k,t}}{(p_k - t)t} v(T) = \frac{p_k}{2(p_k - 1)} v(i) = v(i)$$

and on the other side

$$\frac{m \cdot v(P_k \backslash j)}{(m+1)(p_k - 1)} + \frac{v(i)}{(m+1)(p_k - 1)} = \frac{(m+1)v(i)}{(m+1)(p_k - 1)} = v(i).$$

Case ii) $p_k > 2$ and $|T| = 1$ $(T = i)$ then

$$\frac{p_k}{(p_k - t)t} P^{m,p_k,t} v(i) = \frac{p_k}{(p_k - 1)} P^{m,p_k,1} v(i) = \frac{p_k}{(p_k - 1)} \frac{1}{p_k} \left(1 + \sum_{j=1}^{p_k - 2} \frac{1}{m+j}\right) v(i)$$

and on the other side

$$\frac{v(i)}{(m+1)(p_k - 1)} + \frac{P^{m+1,p_k-1,t}}{t} v(i) = \frac{v(i)}{(m+1)(p_k - 1)} + \frac{1}{p_k - 1} \left(1 + \sum_{j=1}^{p_k - 3} \frac{1}{m+1+j}\right) v(i)$$

.

$$= \frac{1}{p_k - 1} \left(1 + \sum_{j=0}^{p_k - 3} \frac{1}{m+1+j}\right) v(i) = \frac{1}{p_k - 1} \left(1 + \sum_{j=1}^{p_k - 2} \frac{1}{m+j}\right) v(i).$$

Case ii) $p_k > 2$ and $|T| = p_k - 1$ $(T = P_k \backslash j)$ then

$$p_k \frac{P^{m,p_k,t}}{(p_k - t)t} v(P_k \backslash j) = \frac{p_k}{p_k - 1} \frac{m}{(m+1)p_k} v(P_k \backslash j) = \frac{m}{(m+1)(p_k - 1)} v(P_k \backslash j)$$

and on the other side

$$\frac{m \cdot v(P_k \backslash j)}{(m+1)(p_k - 1)}.$$

Case iii) $p_k > 2$ and $|T| = p_k - 2$ then

$$p_k \frac{P^{m,p_k,t}}{2(p_k - 2)} v(T) = \frac{p_k}{2(p_k - 2)} \frac{m+1}{m+2} \frac{1}{p_k - 1} \frac{2}{p_k} v(T) = \frac{1}{(p_k - 2)} \frac{m+1}{m+2} \frac{1}{p_k - 1} v(T)$$

and on the other side

$$\frac{P^{m+1,p_k-1,t}}{t} v(T) = \frac{1}{p_k - 2} \frac{m+1}{m+2} \frac{1}{p_k - 1} v(T).$$

Case iv) $p_k > 2$ and $|T| = p_k - z$ where $z \in \{3, ..., p_k - 2\}$

$$p_k \frac{P^{m,p_k,t}}{(p_k - t)t} v(T) = \frac{p_k}{z(p_k - z)} \frac{m + (z-1)}{(p_k - (z-1))(m+z)} \left(\sum_{l=0}^{z-2} \frac{p_k - l - t}{p_k - l} \right) v(T)$$

$$= \frac{1}{(p_k - z)} \frac{m + (z-1)}{(p_k - (z-1))(m+z)} \left(\sum_{l=1}^{z-2} \frac{p_k - l - t}{p_k - l} \right) v(T)$$

and on the other side

$$\frac{P^{m+1,p_k-1,t}}{t} v(T) = \frac{1}{p_k - z} \frac{m + 1 + (z' - 1)}{(p_k - 1 - (z' - 1))(m + 1 + z')} \left(\sum_{j=0}^{z'-2} \frac{p_k - 1 - j - t}{p_k - 1 - j} \right)$$

where $z' = z - 1$ because $P^{m+1,p_k-1,t}$ depends on $P_k \backslash j$.

Finally, the uniqueness is proven analogously as the uniqueness in Theorem 2 of Vázquez-Brage et al. (1997). $\square$

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