International Journal of Computer Mathematics
Vol. 00, No. 00, December 2013, 1–20

RESEARCH ARTICLE

A new numerical method for pricing Fixed-Rate Mortgages with prepayment and default options

M.C. Calvo-Garrido[a] and C. Vázquez[a*]

[a] *Dep. de Matemáticas, Univ. da Coruña, Campus Elviña s/n, 15071-A Coruña, Spain*

(*December 2013*)

In this paper we consider the valuation of fixed rate mortgages including prepayment and default options, where the underlying stochastic factors are the house price and the interest rate. The mathematical model to obtain the value of the contract is posed as a free boundary problem associated to a PDE model. The equilibrium contract rate is determined by using an iterative process. Moreover, appropriate numerical methods based on a Lagrange-Galerkin discretization of the PDE, an augmented Lagrangian active set method and a Newton iteration scheme are proposed. Finally, some numerical results to illustrate the performance of the numerical schemes, as well as the qualitative and quantitative behaviour of solution and the optimal prepayment boundary are presented.

Keywords: Fixed-rate mortgages; option pricing; complementarity problem; numerical methods; Augmented Lagrangian Active Set formulation

AMS Subject Classification: 91G80, 65M25, 65M60, 90C33

1. Introduction

A mortgage is a financial contract in which the borrower obtains funds (usually from a bank or a financial institution) by using a risky asset (in this case a house) as a collateral. The value of this contract depends on the house price and the interest rate, as underlying factors. In order to pay the mortgage, monthly payments from the borrower to the lender are considered so that cancelation occurs when at the maturity of the loan the debt is totally paid. Thus, the mortgage value is understood as the present value of the borrower scheduled monthly payments without including the insurance the lender can have on the loan. Moreover, in the present paper the possibilities of the remaining mortgage value prepayment and borrower default are also considered. Prepayment can occur at any time during the life of the loan (analogously to the exercise in American options) while default only can happen at any monthly payment date. In fact, at each monthly payment date the borrower decides either to make the payment or default if the house value is less than the mortgage price. Thus, if we consider both prepayment and default option, the pricing problem is equivalent to a sequence of linked American options, one for each month. Moreover, starting from the final mortgage value at last month, the final mortgage value at the end of each month is obtained from the mortgage value for the corresponding next month at the same date.

*Corresponding author. Email: carlosv@udc.es. Paper funded by Spanish MCINN (Project MTM2010–21135–C02-01) and by Xunta de Galicia (Ayuda CN2011/004, partially funded with FEDER funds).

ISSN: 0020-7160 print/ISSN 1029-0265 online

DOI: 0020716YYxxxxxxxx
http://www.informaworld.com

At origination, the contract must be in equilibrium, which is achieved if the value of the mortgage to the lender plus the insurance against default is equal to the amount of money lent to the borrower, otherwise the contract would not be arbitrage free. This equilibrium provides the fixed rate of the loan.

In the literature we can distinguish Fixed-Rate Mortgages (FRM) and Adjustable-Rate mortgages (ARM). In the first case, the interest rate the borrower has to pay is constant while in the second one is floating according to a specific rate index (LIBOR, for example). In this paper we deal with contracts of the first type in which the fixed rate is the equilibrium rate and needs to be adjusted by using an iterative process.

In order to obtain the value of the contract and other components (such as insurance and coinsurance), option pricing methodology can be applied and leads to a sequence of backward in time partial differential equation (PDE). The problem is divided in monthly intervals where the final condition for a given month comes from the value at the same date of following month. Additionally, the option of prepayment leads to free boundary problem formulations.

In [13] the properties of the free boundary are studied for the case in which default is not allowed so that the problem is much simpler as there is only an stochastic factor (the interest rate). Moreover, in order to solve backwards the PDE several numerical methods need to be provided. For example, in [12] and [1] explicit finite-differences schemes have been used. In [17] a semi-implicit Crank-Nicolson finite-difference scheme to discretize the PDE and a projected successive over-relaxation (PSOR) method to solve the complementarity problem (associated to prepayment feature) have been applied. Moreover, a technique based on the application of singular perturbation theory in order to speed up the calculation is also established. Basically, for small volatilities, the higher order terms in the PDE are neglected and the first order PDE is analytically solved. Finally, a comparison between the two methods is presented. However, some differences between the solution of first order PDE and the numerical solution of the original PDE are observed and some comments about the need of using higher order terms in the asymptotic expansion are pointed out, specially in scenarios with higher volatilities. In [19] the inclusion of higher order terms for European and American options is discussed and the corresponding PDE problems require numerical methods of the same complexity of those ones applied to the original problem.

In this paper, we numerically solve the original equation by proposing the PDE discretization with the techniques developed in [4] for Asian options and more recently applied to pension plans in [5] and [6]. More precisely, we use a characteristics method to discretize first order terms and a Crank-Nicolson scheme that evaluates the functions at the previous time step in the basis of the characteristics, which consists on a different approach from the one proposed in [17]. These methods are particularly well suited for convection dominated problems, as those ones appearing in the case of small volatilities. If we neglect second order terms, then we recover the perturbation based solution proposed in [17]. The numerical analysis of the proposed characteristics Crank-Nicolson time discretization, the fully discretized problem when combined with Lagrange finite elements and the use of numerical integration formulas has been addressed in [2] and [3]. Both papers are applied to general convection-diffusion-reaction equations under certain assumptions. Furthermore, the non-linearities associated with the inequality constraints in the complementarity formulation due to prepayment are treated by means of the recently introduced Augmented Lagrangian Active Set (ALAS) method [11].

The paper is organized as follows. In Section 2, first we state the mathematical model by describing the stochastic variables and deriving the PDE that governs the valuation of the mortgage components. Then, we establish the final, payment date and equilibrium conditions, as well as other characteristics of the contract. Also, the free boundary problem associated with prepayment option is presented. Section 3 contains the description of the numerical techniques. Some numerical results are presented in Section 4. Finally, some conclusions are discussed in Section 5.

2. Mathematical modeling

2.1 *Stochastic initial financial framework*

A mortgage can be treated as a derivative financial product, for which the underlying state variables are the house price and the term structure of interest rates.

The value of the house at time t, H_t, is assumed to follow the standard log-normal process (see [14]), that satisfies the following stochastic differential equation:

$$dH_t = (\mu - \delta)H_t dt + \sigma_H H_t dX_t^H, \tag{1}$$

where

- μ is the instantaneous average rate of house-price appreciation,
- δ is the 'dividend-type' per unit service flow provided by the house,
- σ_H is the house-price volatility,
- and X_t^H is the standardized Wiener process for house price.

Note that this process has an absorbing barrier at zero, meaning that if H_t reaches at any time the value zero, it remains zero thereafter. The dividend-type parameter δ is associated to the benefits of owning the house (usage, hiring, ...). The previous model does not take into account possible jumps in the house price, which would require the use of jump-diffusion models.

Deriving the risk-neutral process for house price by changing to a risk neutral probability measure involves replacing the expected drift term $\mu-\delta$ by $\mu-\delta-\lambda\sigma$, where λ represents the market price of risk associated to the uncertainty of the house price [8]. Using risk neutrality arguments, $\mu - \lambda\sigma$ is equal to the risk-free interest rate r_t. So, by substituting this expression in equation (1), we obtain

$$dH_t = (r_t - \delta)H_t dt + \sigma_H H_t dX_t^H. \tag{2}$$

The other source of uncertainty, the interest rate r_t at time t, is assumed to be stochastic and its evolution can be modeled with the following classical Cox-Ingersoll-Ross (CIR) process [7],

$$dr_t = \kappa(\theta - r_t)dt + \sigma_r \sqrt{r_t} dX_t^r, \tag{3}$$

where

- κ is the speed of adjustment in the mean reverting process,
- θ is the long term mean of the short-term interest rate (steady state spot rate),
- σ_r is the interest-rate volatility parameter,
- and X_t^r is the standardized Wiener process for interest rate.

Notice that the CIR model is mean-reverting. Moreover, if $2\kappa\theta \geq \sigma_r^2$ and $r_0 > 0$ then zero is a natural reflecting barrier and negative interest rates cannot be achieved. In [13] a Vasicek model is considered so that negative interest rates can be obtained.

Wiener processes, X_t^H and X_t^r can be assumed to be correlated according to $dX_t^H dX_t^r = \rho dt$, where ρ is the instantaneous correlation coefficient.

2.2 *Statement of the morgage pricing PDE problem*

The price of any asset whose value is a function of house price H_t, interest rate r_t and time t is a stochastic process, $F_t = F(t, H_t, r_t)$, where F is a smooth enough function. Then, by using the dynamic hedging methodology [16], the function F is the solution of a certain PDE problem. Here, it is assumed that the house price evolution is described by equation (2) and the interest rate dynamics is governed by equation (3). So, we can apply Itô's Lemma (see [9], for example) to obtain the variation of F_t, dF_t, from time t to $t + dt$ for small dt. Hereafter, we suppress the dependence on t in order to simplify notation:

$$dF = \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial H}dH + \frac{\partial F}{\partial r}dr + \frac{1}{2}\left(\sigma_H^2 H^2 \frac{\partial^2 F}{\partial H^2} + 2\rho\sigma_H\sigma_r H\sqrt{r}\frac{\partial^2 F}{\partial H \partial r} + \sigma_r^2 r \frac{\partial^2 F}{\partial r^2}\right)dt \tag{4}$$

At this point, we construct a portfolio Π by buying one unit of the asset F_1 with maturity T_1 and selling Δ_2 and Δ_1 units of the asset F_2 with maturity T_2 and of the underlying asset H, respectively. Thus,

$$\Pi = F_1 - \Delta_2 F_2 - \Delta_1 H \tag{5}$$

Note that the variation of the portfolio value between t and $t + dt$ is given by:

$$d\Pi = dF_1 - \Delta_2 dF_2 - \Delta_1 dH, \tag{6}$$

where Δ_1 and Δ_2 are constant in $[t, t+dt]$. As it is the case of dividends in options on assets, the effect of the service flow δ causes the price of the underlying asset H to drop in value by δH over a time interval $[t, t + dt]$. Therefore, the portfolio must change by an amount $-\delta H \Delta_1 dt$ during this time interval. Thus, the correct change in the value of the portfolio is

$$d\Pi = dF_1 - \Delta_2 dF_2 - \Delta_1(dH + \delta H dt). \tag{7}$$

Moreover, Π turns out to be risk-free for the following choice:

$$\Delta_2 = \frac{\partial F_1/\partial r}{\partial F_2/\partial r}, \quad \Delta_1 = \frac{\partial F_1}{\partial H} - \Delta_2 \frac{\partial F_2}{\partial H} \tag{8}$$

So, for this choice of Δ, the variation of the risk-free portfolio is given by:

$$\begin{aligned} d\Pi = \Bigg[& \frac{\partial F_1}{\partial t} + \frac{1}{2}\left(\sigma_H^2 H^2 \frac{\partial^2 F_1}{\partial H^2} + 2\rho\sigma_H\sigma_r H\sqrt{r}\frac{\partial^2 F_1}{\partial H \partial r} + \sigma_r^2 r \frac{\partial^2 F_1}{\partial r^2}\right) - \delta H \frac{\partial F_1}{dH} \\ & - \frac{\partial F_1/\partial r}{\partial F_2/\partial r}\left(\frac{\partial F_2}{\partial t} + \frac{1}{2}\left(\sigma_H^2 H^2 \frac{\partial^2 F_2}{\partial H^2} + 2\rho\sigma_H\sigma_r H\sqrt{r}\frac{\partial^2 F_2}{\partial H \partial r} + \sigma_r^2 r \frac{\partial^2 F_2}{\partial r^2}\right) - \delta H \frac{\partial F_2}{dH}\right)\Bigg] dt. \end{aligned}$$

By using the arbitrage-free assumption, this variation is also given by $d\Pi = r\Pi dt$. Thus, we obtain the identity:

$$\frac{1}{\partial F_1/\partial r}\left(\frac{\partial F_1}{\partial t} + \frac{1}{2}\sigma_H^2 H^2 \frac{\partial^2 F_1}{\partial H^2} + \rho\sigma_H\sigma_r H\sqrt{r}\frac{\partial^2 F_1}{\partial H\partial r} + \frac{1}{2}\sigma_r^2 r \frac{\partial^2 F_1}{\partial r^2} + (r-\delta)H\frac{\partial F_1}{\partial H} - rF_1\right)$$
$$= \frac{1}{\partial F_2/\partial r}\left(\frac{\partial F_2}{\partial t} + \frac{1}{2}\sigma_H^2 H^2 \frac{\partial^2 F_2}{\partial H^2} + \rho\sigma_H\sigma_r H\sqrt{r}\frac{\partial^2 F_2}{\partial H\partial r} + \frac{1}{2}\sigma_r^2 r \frac{\partial^2 F_2}{\partial r^2} + (r-\delta)H\frac{\partial F_2}{\partial H} - rF_2\right).$$

The left hand side of the equality is a function of T_1 but not of T_2 and the right side is a function of T_2 but not T_1. This is only possible if both sides are independent of maturity date, so that

$$\frac{1}{\partial F/\partial r}\left(\frac{\partial F}{\partial t} + \frac{1}{2}\sigma_H^2 H^2 \frac{\partial^2 F}{\partial H^2} + \rho\sigma_H\sigma_r H\sqrt{r}\frac{\partial^2 F}{\partial H\partial r} + \right.$$
$$\left. + \frac{1}{2}\sigma_r^2 r \frac{\partial^2 F}{\partial r^2} + (r-\delta)H\frac{\partial F}{\partial H} - rF\right) = a(t,H,r), \tag{9}$$

where it is convenient to write $a(t,H,r) = -\kappa(\theta - r)$, which is a standard procedure in the literature (see [12], [1], for example).

So, by reordering the terms in (9) we obtain the following PDE that governs the valuation of any asset depending on house price and interest rate, in particular the fixed-rate mortgages.

$$\frac{\partial F}{\partial t} + \frac{1}{2}\sigma_H^2 H^2 \frac{\partial^2 F}{\partial H^2} + \rho\sigma_H\sigma_r H\sqrt{r}\frac{\partial^2 F}{\partial H\partial r} + \frac{1}{2}\sigma_r^2 r \frac{\partial^2 F}{\partial r^2} +$$
$$+ (r-\delta)H\frac{\partial F}{\partial H} + \kappa(\theta - r)\frac{\partial F}{\partial r} - rF = 0. \tag{10}$$

2.3 *Mortgage contract*

In the fixed-rate mortgage we are considering, the loan is repaid by a series of equal monthly payments at given dates T_m, $m = 1, \ldots, M$. Moreover, assuming $T_0 = 0$, let $\Delta T_m = T_m - T_{m-1}$ denote the duration of month m. Thus, assuming that M is the number of months, c is the fixed contract rate and $P(0)$ is the initial loan (i.e. the principal at $t = T_0 = 0$), the fixed mortgage payment (MP) is given by formula:

$$MP = \frac{(c/12)(1+c/12)^M P(0)}{(1+c/12)^M - 1}, \tag{11}$$

For $m = 1, \ldots, M$, the unpaid loan just after the $(m-1)$th payment is given by

$$P(m-1) = \frac{((1+c/12)^M - (1+c/12)^{m-1})P(0)}{(1+c/12)^M - 1}, \tag{12}$$

If $t_m = t - T_{m-1}$ denotes the time elapsed at month m (which starts at $t = T_{m-1}$), we introduce $\tau_m = \Delta T_m - t_m$ as the time until T_m. This change of time variable transforms equation (10) into another one associated to an initial value problem. More precisely, the mortgage value to the lender during month m, $V(\tau_m, H, r)$, without including the insurance the lender has on the loan, satisfies the PDE

$$\frac{\partial F}{\partial \tau_m} - \frac{1}{2}\sigma_H^2 H^2 \frac{\partial^2 F}{\partial H^2} - \rho\sigma_H\sigma_r H\sqrt{r}\frac{\partial^2 F}{\partial H \partial r} - \frac{1}{2}\sigma_r^2 r \frac{\partial^2 F}{\partial r^2} - \\ -(r-\delta)H\frac{\partial F}{\partial H} - \kappa(\theta - r)\frac{\partial F}{\partial r} + rF = 0, \qquad (13)$$

for $0 \le \tau_m \le \Delta T_m$, $0 \le H < \infty$, $0 \le r < \infty$. We clarify a certain abuse of notation: if $\overline{F}$ denotes the solution of (10) and F the solution of (13) then $F(\tau_m, H, r) = \overline{F}(T_m - \tau_m, H, r)$, .

In the mortgage contract we consider there are two embedded options for the borrower. On one hand the option to default on the mortgage that can only happen at payment dates once the borrower decides not to pay the monthly amount MP, and on the other hand the option to prepay the mortgage, which can be exercised at any time during the life of the loan. If the borrower decides to fully amortize the mortgage at time τ_m, he/she should pay the total debt payment $TD(\tau_m)$, which includes an early termination penalty and is given by expression

$$TD(\tau_m) = (1+\Psi)(1 + c(\Delta T_m - \tau_m))P(m-1), \qquad (14)$$

where Ψ denotes the prepayment penalty factor.

Thus, at each payment date the borrower must decide whether to pay the required monthly payment or default and hand over the house to the lender. The option to prepay gives the borrower the right to exercise the prepayment at any time during the lifetime of the mortgage (American feature).

The mortgage pricing problem starts from the value of the mortgage at maturity $(t = T_M)$, which just before the last payment is given by

$$V(\tau_M = 0, H, r) = \min(MP, H) \qquad (15)$$

while at the other payment dates, it is given by

$$V(\tau_m = 0, H, r) = \min(V(\tau_{m+1} = \Delta T_{m+1}, H, r) + MP, H), \qquad (16)$$

where $1 \le m \le M-1$.

If the borrower defaults, which occurs when the mortgage value is equal to the house value, the lender will lose the promised future payments. Then, the lender might have taken an insurance against default which would cover a fraction of the loss associated to default. As indicated in [17] this asset adds to the lender's position in the contract. In order to obtain the value of this insurance to the lender, denoted by $I(\tau_m, H, r)$, we must solve equation (13) with suitable payment date conditions. In order to pose them, we assume that in case of default the insurer accepts to pay a fraction γ of the currently unpaid balance to the lender up to a maximum indemnity or cap, Γ. By taking this into account, depending if default occurs or not, the insurance value at the maturity of the loan is

$$I(\tau_M = 0, H, r) = \begin{cases} \min(\gamma(MP - H), \Gamma) & \text{(Default)} \\ 0 & \text{(No default)} \end{cases} \qquad (17)$$

At earlier payment dates, in case of default the value of the insurance is

$$I(\tau_m = 0, H, r) = \begin{cases} \min(\gamma[TD(\tau_m = 0) - H], \Gamma) & \text{(Default)} \\ I(\tau_{m+1} = \Delta T_{m+1}, H, r) & \text{(No default)} \end{cases} \qquad (18)$$

where $1 \leq m \leq M-1$.

The fraction of the potential loss not covered by the insurance is referred as the coinsurance. At each payment date, the coinsurance is the difference between the values of the potential loss and the insurance coverage. In this case, in order to price the coinsurance, , $CI(\tau_m, H, r)$, equation (13) must be solved again with suitable conditions. At maturity, the value of the coinsurance is

$$CI(\tau_M = 0, H, r) = \begin{cases} \max((1-\gamma)(MP-H), (MP-H)-\Gamma) & \text{(Default)} \\ 0 & \text{(No default)} \end{cases} \tag{19}$$

At earlier payment dates, the value of the coinsurance is

$$CI(\tau_m = 0, H, r) = \begin{cases} \max((1-\gamma)[TD(\tau_m = 0) - H], [TD(\tau_m = 0) - H] - \Gamma) & \text{(Default)} \\ CI(\tau_{m+1} = \Delta T_{m+1}, H, r) & \text{(No default)} \end{cases} \tag{20}$$

where $1 \leq m \leq M-1$.

2.4 *Arbitrage free condition*

At the time of origination, the value of the contract together with the insurance and any upfront points must be the same to the lender as the value of the loan to the borrower. Thus, arbitrage is avoided and the contract is fair for both parts. Formally,

$$V(\tau_1 = \Delta T_1, H_{initial}, r_{initial}; \Psi, c) + I(\tau_1 = \Delta T_1, H_{initial}, r_{initial}; \Psi, c) = (1-\xi)P(0), \tag{21}$$

where $\xi P(0)$ is the value of the upfront points, understood as an arrangement fee. The arrangement fee, the prepayment penalty Ψ and whether or not the lender holds an insurance are specified in the contract. So, this equation contains only one free parameters, the contract rate c. It is necessary to find the value of the interest rate c which satisfies the equilibrium condition (21) and ensures that the contract is fair and arbitrage free. It can be obtained by using an iterative method for nonlinear equations.

2.4.1 Arbitrage equilibrium analysis

In order to give an idea of the equilibrium mortgage contract rate, different contracts are considered (see [12]):

- Basic contract: in this simple case the arrangement fee $\xi = 0$ and no insurance is charged. So, equation (21) reduces to

$$V(\tau_1 = \Delta T_1, H_{initial}, r_{initial}; \Psi, c) = P(0). \tag{22}$$

The arbitrage condition requires that $(H_{initial}, r_{initial})$ be a point in state space where immediate prepayment is an optimal strategy. For all values of $c > \hat{c}$ the point $(H_{initial}, r_{initial})$ is in fact in the interior of the prepayment region. Since the borrower simultaneously takes the loan and pays it off on the right of $\hat{c}$, no equilibrium is observed when $c > \hat{c}$ and $\hat{c}$ is not really a valid solution because the borrower is indifferent between prepayment and continuation. The practise of loaning less than the full value of the house in order to reduce the risk of the loan is a standard one. In our case when $P(0) = H$ no equilibrium could exists, since it implies that default would also be an optimal strategy and it is not possible because the borrower could earn the flow of service on the house until the first payment becomes due.

- Contract with points: if an arrangement fee, ξ, is introduced into the equation, the equilibrium equation has this expression:

$$V(\tau_1 = \Delta T_1, H_{initial}, r_{initial}; \Psi, c) = (1 - \xi)P(0) \tag{23}$$

  In this case, the equilibrium contract rate c is $c_1 < \hat{c}$. Then, the problem of a continuum of values satisfying equation (22) is removed. Now, the point $(H_{initial}, r_{initial})$ is in the interior of the continuation region.
- Contract with insurance: now we consider the case where insurance can have value, but upfront points are no charged ($\xi = 0$). The expression for the equilibrium condition in this case is as follows:

$$V(\tau_1 = \Delta T_1, H_{initial}, r_{initial}; \Psi, c) + I(\tau_1 = \Delta T_1, H_{initial}, r_{initial}; \Psi, c) = P(0) \tag{24}$$

  Now, there is an isolated equilibrium when $c = c_2$ such that, $c_1 < c_2 < \hat{c}$ as well as the continuum, $c \geq \hat{c}$. At these latter value immediate prepayment is the optimal strategy, so insurance has no value, and in the other case at c_2 insurance has positive value.
- Full contract: this is the general case with insurance and upfront points. The equilibrium equation is this case is (21). There is an unique value of $c = c_3$ which satisfies the equation. Therefore, it is necessary that $c_3 \leq c_1$ and $c_3 \leq c_2$.

2.5 *The free boundary problem*

Let us consider the following linear operator,

$$\begin{aligned}\mathcal{L}V \equiv \frac{\partial V}{\partial \tau_m} - \frac{1}{2}\sigma_H^2 H^2 \frac{\partial^2 V}{\partial H^2} - \rho\sigma_H\sigma_r H\sqrt{r}\frac{\partial^2 V}{\partial H \partial r} - \frac{1}{2}\sigma_r^2 r\frac{\partial^2 V}{\partial r^2}\\ -(r-\delta)H\frac{\partial V}{\partial H} - \kappa(\theta - r)\frac{\partial V}{\partial r} + rV.\end{aligned} \tag{25}$$

So, the free boundary problem associated with the valuation of the mortgage contract, can be reduced to the linear complementarity problem:

$$\mathcal{L}V \leq 0, \quad (TD(\tau_m) - V(\tau_m, H, r)) \geq 0, \quad (\mathcal{L}V)(TD(\tau_m) - V(\tau_m, H, r)) = 0. \tag{26}$$

The option to prepay can be exercised at any time during the lifetime of the contract. If $V = TD$ then it is optimal for the borrower to prepay, otherwise $\mathcal{L}V = 0$ and it is optimal to maintain the loan.

3. Numerical methods

In order to obtain a numerical approach of the value of the contract at origination, we need to solve a free boundary problem for each month to obtain the value of the mortgage during that month, jointly with an additional initial value problem when the lender holds an insurance. Once we know the value at origination of the contract and the insurance, the equilibrium condition (21) is checked to find the interest rate for which the contract is arbitrage free. For this purpose, a Newton-like method is implemented. By using the equilibrium rate, we solve another initial value problem to obtain the coinsurance. For the numerical solution of the PDE, we propose a Crank-Nicolson characteristics time discretization scheme combined with quadratic

Lagrange finite element method. Thus, first a localization technique is used to cope with the initial formulation in an unbounded domain. For the inequality constraints associated with the complementarity problem, we propose a mixed formulation and an augmented Lagrangian active set technique.

3.1 *Localization procedure and formulation in a bounded domain*

In this section we replace the unbounded domain by a bounded one and determine the required boundary conditions. For this purpose, we introduce the notation:

$$x_0 = \tau_m, \quad x_1 = \frac{H}{H_\infty} \quad \text{and} \quad x_2 = \frac{r}{r_\infty}, \tag{27}$$

where both H_∞ and r_∞ are sufficiently large suitably chosen real numbers. Let $\Omega = (0, x_0^\infty) \times (0, x_1^\infty) \times (0, x_2^\infty)$, with $x_0^\infty = \Delta T_m$, $x_1^\infty = x_2^\infty = 1$. Then, let us denote the Lipschitz boundary by $\Gamma = \partial\Omega$ such that $\Gamma = \bigcup_{i=0}^{2}(\Gamma_i^- \cup \Gamma_i^+)$, where:

$$\Gamma_i^- = \{(x_0, x_1, x_2) \in \Gamma \mid x_i = 0\}, \ \Gamma_i^+ = \{(x_0, x_1, x_2) \in \Gamma \mid x_i = x_i^\infty\}, \ i = 0, 1, 2.$$

Then, the PDE in problem (13) can be written in the form:

$$\sum_{i,j=0}^{2} b_{ij} \frac{\partial^2 F}{\partial x_i x_j} + \sum_{j=0}^{2} b_j \frac{\partial F}{\partial x_j} + b_0 F = f_0, \tag{28}$$

where the involved data are defined as follows:

$$B = (b_{ij}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2}\sigma_H^2 x_1^2 & \frac{1}{2}\rho x_1 \sqrt{x_2/r_\infty}\sigma_H \sigma_r \\ 0 & \frac{1}{2}\rho x_1 \sqrt{x_2/r_\infty}\sigma_H \sigma_r & \frac{1}{2}\sigma_r^2 x_2 / r_\infty \end{pmatrix}, \tag{29}$$

$$\vec{b} = (b_j) = \begin{pmatrix} -1 \\ (x_2 r_\infty - \delta) x_1 \\ \kappa(\theta - x_2 r_\infty)/r_\infty \end{pmatrix}, \quad b_0 = -x_2 r_\infty, \quad f_0 = 0. \tag{30}$$

Thus, following [15], in terms of the normal vector to the boundary pointing inward Ω, $\vec{m} = (m_0, m_1, m_2)$, we introduce the following subsets of Γ:

$$\Sigma^0 = \left\{ x \in \Gamma / \sum_{i,j=0}^{2} b_{ij} m_i m_j = 0 \right\}, \quad \Sigma^1 = \Gamma - \Sigma^0,$$

$$\Sigma^2 = \left\{ x \in \Sigma^0 / \sum_{i=0}^{2} \left(b_i - \sum_{j=0}^{2} \frac{\partial b_{ij}}{\partial x_j} \right) m_i < 0 \right\}.$$

As indicated in [15] the boundary conditions at $\Sigma^1 \bigcup \Sigma^2$ for the so-called first boundary value problem associated with (28) are required. Note that $\Sigma^1 = \Gamma_1^+ \bigcup \Gamma_2^+$ and $\Sigma^2 = \Gamma_0^-$. Therefore, in addition to an initial condition depending on the payment date Γ_0^- (see section 2.3), we impose the following Neumann conditions:

$$\frac{\partial F}{\partial x_1} = 0 \quad on \quad \Gamma_1^+, \tag{31}$$

$$\frac{\partial F}{\partial x_2} = 0 \quad on \quad \Gamma_2^+. \tag{32}$$

Next, taking into account the new variables we write the equation (13) in divergence form in the bounded domain. As in [17], we consider the case $\rho = 0$. Thus, the initial-boundary value problem for the insurance and coinsurance can be written in the form: Find $J : [0, \Delta T_m] \times \Omega \to \mathbb{R}$ such that

$$\frac{\partial J}{\partial \tau_m} + \vec{v} \cdot \nabla J - div(A \nabla J) + lJ = f \ \text{ in } (0, \Delta T_m) \times \Omega\,, \tag{33}$$

$$\frac{\partial J}{\partial x_1} = g_1 \ \text{ on } (0, \Delta T_m) \times \Gamma_1^+\,, \tag{34}$$

$$\frac{\partial J}{\partial x_2} = g_2 \ \text{ on } (0, \Delta T_m) \times \Gamma_2^+\,, \tag{35}$$

where $J = I, CI$ and the appropriate initial condition for each month is given by the equations (17) and (18) when we are pricing the insurance and by the equations (19) and (20) in the case of valuing the coinsurance.

Furthermore, for the complementarity problem associated to the mortgage value during montn m, we can pose the following mixed formulation:

Find $V : [0, \Delta T_m] \times \Omega \to \mathbb{R}$ satisfying the partial differential equation

$$\frac{\partial V}{\partial \tau_m} + \vec{v} \cdot \nabla V - div(A \nabla V) + lV + P = f \ \text{ in } (0, \Delta T_m) \times \Omega\,, \tag{36}$$

the complementarity conditions

$$V \leq TD, \quad P \geq 0, \quad P(TD - V) = 0 \ \text{ in } (0, \Delta T_m) \times \Omega \tag{37}$$

the boundary conditions

$$\frac{\partial V}{\partial x_1} = g_1 \ \text{ on } (0, \Delta T_m) \times \Gamma_1^+\,, \tag{38}$$

$$\frac{\partial V}{\partial x_2} = g_2 \ \text{ on } (0, \Delta T_m) \times \Gamma_2^+ \tag{39}$$

and the initial condition for each month, given by the equations (15) or (16).

For both problems, the involved data is defined as follows

$$A = \begin{pmatrix} \frac{1}{2}\sigma_H^2 x_1^2 & 0 \\ 0 & \frac{1}{2}\sigma_r^2 \frac{x_2}{r_\infty} \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} (\sigma_H^2 - x_2 r_\infty + \delta) x_1 \\ (\frac{1}{2}\sigma_r^2 - \kappa(\theta - x_2 r_\infty))/r_\infty \end{pmatrix} \tag{40}$$

$$l = x_2 r_\infty, \quad f = 0,\, g_1 = 0,\, g_2 = 0. \tag{41}$$

Next, the qualitative behaviour of the velocity field on the boundaries is studied:

- On boundary Γ_1^-, since $x_1 = 0$ then

$$\vec{v} = \left(0, \left(\frac{1}{2}\sigma_r^2 - \kappa(\theta - x_2 r_\infty)\right)/r_\infty\right),$$

so the velocity field is tangential to the boundary.

- On boundary Γ_2^-, since $x_2 = 0$ then

$$\vec{v} = \left((\sigma_H + \delta)x_1, \left(\frac{1}{2}\sigma_r^2 - \kappa\theta\right)/r_\infty\right),$$

so as $\sigma_r \leq \sqrt{2\kappa\theta}$ the velocity field either points outward the domain or it is tangential to the boundary.

- On boundary Γ_1^+, since $x_1 = 1$ then

$$\vec{v} = \left(\sigma_H - r_\infty x_2 + \delta, \left(\frac{1}{2}\sigma_r^2 - \kappa(\theta - x_2 r_\infty)\right)/r_\infty\right),$$

so if $(\sigma_H^2 + \delta) < r_\infty x_2$ the velocity field enters the domain, otherwise it points outward the domain.

- On boundary Γ_2^+, since $x_2 = 1$ then

$$\vec{v} = \left((\sigma_H - r_\infty + \delta)x_1, \left(\frac{1}{2}\sigma_r^2 - \kappa(\theta - r_\infty)\right)/r_\infty\right),$$

so if $\frac{1}{2}\sigma_r^2 < \kappa(\theta - r_\infty)$ the velocity field enters the domain, otherwise it points outward the domain.

3.2 *Time discretization*

First, we define the characteristics curve through $\mathbf{x} = (x_1, x_2)$ at time $\bar{\tau}_m$, $X(\mathbf{x}, \bar{\tau}_m; s)$, which satisfies:

$$\frac{\partial}{\partial s} X(\mathbf{x}, \bar{\tau}_m; s) = \vec{v}(X(\mathbf{x}, \bar{\tau}_m; s)),\ X(\mathbf{x}, \bar{\tau}_m; \bar{\tau}_m) = \mathbf{x}. \tag{42}$$

For $N > 1$ let us consider the time step $\Delta\tau_m = \Delta T_m / N$ and the time mesh points $\tau_m^n = n\Delta\tau_m$, $n = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots, N$. The material derivative approximation by characteristics method is given by:

$$\frac{DF}{D\tau_m} = \frac{F^{n+1} - F^n \circ X^n}{\Delta\tau_m},$$

where $F = CI,\ I, V$ and $X^n(\mathbf{x}) := X(\mathbf{x}, \tau_m^{n+1}; \tau_m^n)$. In view of the expression of the velocity field the components of $X^n(\mathbf{x})$ can be analytically computed:

$$
\begin{aligned}
X_1^n(\mathbf{x}) &= x_1 \exp\left(-\left(\sigma_H^2+\delta+\frac{\sigma_r^2}{2\kappa}-\theta\right)\Delta\tau_m\right)\times \\
&\quad \exp\left(\left(\frac{-x_2 r_\infty}{\kappa}-\frac{\sigma_r^2}{2\kappa^2}+\frac{\theta}{\kappa}\right)(\exp(-\kappa\Delta\tau_m)-1)\right) \\
X_2^n(\mathbf{x}) &= \left(-\frac{\sigma_r^2}{2\kappa r_\infty}+\frac{\theta}{r_\infty}\right)(1-\exp(-\kappa\Delta\tau_m))+x_2\exp(-\kappa\Delta\tau_m)
\end{aligned}
$$

Next, we consider a Crank-Nicolson scheme around $\left(X(\mathbf{x},\tau_m^{n+1};\tau_m),\tau_m\right)$ for $\tau_m = \tau_m^{n+\frac{1}{2}}$. So, for $n = 0,\ldots,N-1$, the time discretized equation for $F = I$, CI, V and $P = 0$ can be written as follows:

Find F^{n+1} such that:

$$
\begin{aligned}
\frac{F^{n+1}(\mathbf{x})-F^n(X^n(\mathbf{x}))}{\Delta\tau_m} &- \frac{1}{2}div(A\nabla F^{n+1})(\mathbf{x}) - \frac{1}{2}div(A\nabla F^n)(X^n(\mathbf{x})) + \\
&\frac{1}{2}(l\,F^{n+1})(\mathbf{x}) + \frac{1}{2}(l\,F^n)(X^n(\mathbf{x})) = 0
\end{aligned}
\tag{43}
$$

In order to obtain the variational formulation of the semi-discretized problem, we multiply equation (43) by a suitable test function, integrate in Ω, use the classical Green formula and the following one ([4]):

$$
\begin{aligned}
\int_\Omega div(\mathbf{A}\nabla F^n)(X^n(\mathbf{x}))\Psi(\mathbf{x})d\mathbf{x} &= \int_\Gamma (\nabla X^n)^{-T}(\mathbf{x})\mathbf{n}(x)\cdot(\mathbf{A}\nabla F^n)(X^n(\mathbf{x}))\Psi(\mathbf{x})d\mathbf{x} \\
&\quad - \int_\Omega (\nabla X^n)^{-1}(\mathbf{x})(\mathbf{A}\nabla F^n)(X^n(\mathbf{x}))\cdot\nabla\Psi(\mathbf{x})d\mathbf{x} \\
&\quad - \int_\Omega div((\nabla X^n)^{-T}(\mathbf{x}))(\mathbf{A}\nabla F^n)(X^n(\mathbf{x}))\Psi(\mathbf{x})d\mathbf{x}
\end{aligned}
\tag{44}
$$

Note that, in the present case, we have:

$$
div((\nabla X^n)^{-T}(\mathbf{x})) = \begin{pmatrix} 0 \\ \frac{r_\infty}{\kappa}(1-\exp(\kappa\Delta\tau_m)) \end{pmatrix}. \tag{45}
$$

After these steps, we can write a variational formulation for the semi-discretized problem as follows:

Find $F^{n+1} \in H^1(\Omega)$ such that, for all $\Psi \in H^1(\Omega)$:

$$\begin{aligned}
&\int_\Omega F^{n+1}(\mathbf{x})\Psi(\mathbf{x})d\mathbf{x} + \frac{\Delta\tau_m}{2}\int_\Omega (\mathbf{A}\nabla F^{n+1})(\mathbf{x})\nabla\Psi(\mathbf{x})d\mathbf{x} + \\
&\quad + \frac{\Delta\tau_m}{2}\int_\Omega lF^{n+1}(\mathbf{x})\Psi(\mathbf{x})d\mathbf{x} = \\
&\int_\Omega F^n(X^n(\mathbf{x}))\Psi(\mathbf{x})d\mathbf{x} - \frac{\Delta\tau_m}{2}\int_\Omega (\nabla X^n)^{-1}(\mathbf{x})(\mathbf{A}\nabla F^n)(X^n(\mathbf{x}))\nabla\Psi(\mathbf{x})d\mathbf{x} - \\
&\quad - \frac{\Delta\tau_m}{2}\int_\Omega lF^n(X^n(\mathbf{x}))\Psi(\mathbf{x})d\mathbf{x} + \frac{\Delta\tau_m}{2}\int_\Gamma \tilde{g}^n(\mathbf{x})\Psi(\mathbf{x})dA_{\mathbf{x}} + \\
&\quad + \frac{\Delta\tau_m}{2}\int_{\Gamma_{1+}} \bar{g}_1^{n+1}(\mathbf{x})\Psi(\mathbf{x})dA_{\mathbf{x}} + \frac{\Delta\tau_m}{2}\int_{\Gamma_{2+}} \bar{g}_2^{n+1}(\mathbf{x})\Psi(\mathbf{x})dA_{\mathbf{x}} - \\
&\quad - \frac{\Delta\tau_m}{2}\int_\Omega div((\nabla X^n)^{-T}(\mathbf{x}))(\mathbf{A}\nabla F^n)(X^n(\mathbf{x}))\Psi(\mathbf{x})d\mathbf{x}
\end{aligned} \tag{46}$$

where ∇X^n can be analytically computed, $\bar{g}_1(\mathbf{x}) = g_1(\mathbf{x})a_{11}(\mathbf{x}) = 0$, $\bar{g}_2(\mathbf{x}) = g_2(\mathbf{x})a_{22}(\mathbf{x}) = 0$ and

$$\tilde{g}^n(\mathbf{x}) := \begin{cases}
-\left[(\nabla X^n)^{-T}\right]_{21}(\mathbf{x})a_{22}(X^n(\mathbf{x}))\frac{\partial F}{\partial x_2}(X^n(\mathbf{x})) & on\ \Gamma_1^- \\
0 & on\ \Gamma_2^- \\
\left[(\nabla X^n)^{-T}\right]_{22}(\mathbf{x})a_{22}(X^n(\mathbf{x}))g_2^n(X^n(\mathbf{x})) & on\ \Gamma_2^+ \\
\left[(\nabla X^n)^{-T}\right]_{11}(\mathbf{x})a_{11}(X^n(\mathbf{x}))g_1^n(X^n(\mathbf{x})) + & \\
+\left[(\nabla X^n)^{-T}\right]_{21}(\mathbf{x})a_{22}(X^n(\mathbf{x}))\frac{\partial F}{\partial x_2}(X^n(\mathbf{x})) & on\ \Gamma_1^+
\end{cases} \tag{47}$$

3.3 *Finite elements discretization*

For the spatial discretization we consider $\{\tau_h\}$ a quadrangular mesh of the domain Ω. Let $(T, \mathcal{Q}_2, \Sigma_T)$ be a family of piecewise quadratic Lagrangian finite elements, where $\mathcal{Q}_2$ is the space of polynomials defined in $T \in \tau_h$ with degree less or equal than two in each spatial variable and Σ_T the subset of nodes of the element T. More precisely, let us define the finite elements space F_h by

$$V_h = \{\phi_h \in \mathcal{C}^0(\bar{\Omega}) : \phi_{h_T} \in \mathcal{Q}_2, \forall T \in \tau_h\}, \tag{48}$$

where $\mathcal{C}^0(\bar{\Omega})$ is the space of piecewise continuous functions on $\bar{\Omega}$.

3.4 *Augmented Lagrangian Active Set (ALAS) algorithm*

The Augmented Lagrangian Active Set (ALAS) algorithm proposed in [11] is here applied to the fully discretized in time and space mixed formulation (36)-(37). More precisely, after this discretization, the discrete problem can be written in the form:

$$M_h V_h^n + P_h^n = b_h^{n-1}, \tag{49}$$

with the discrete complementarity conditions

$$V_h^n \leq TD_h^n, \quad P_h^n \geq 0, \quad (TD_h^n - V_h^n)\, P_h^n = 0, \tag{50}$$

where P_h^n denotes the vector of the multiplier values and TD_h^n denotes the vector of the nodal values defined by function TD.

The basic iteration of the ALAS algorithm consists of two steps. In the first one the domain is decomposed into active and inactive parts (depending on whether the constraints are active or not), and in the second step a reduced linear system associated with the inactive part is solved. We use the algorithm for unilateral problems, which is based on the augmented Lagrangian formulation.

First, for any decomposition $\mathcal{N} = \mathcal{I} \cup \mathcal{J}$, where $\mathcal{N} := \{1, 2, \ldots N_{dof}\}$, let us denote by $[M_h]_{\mathcal{I}\mathcal{I}}$ the principal minor of matrix M_h and by $[M_h]_{\mathcal{I}\mathcal{J}}$ the co-diagonal block indexed by $\mathcal{I}$ and $\mathcal{J}$. Thus, for each mesh time τ_{m_n}, the ALAS algorithm computes not only V_h^n and P_h^n but also a decomposition $N = \mathcal{J}^n \cup \mathcal{I}^n$ such that

$$\begin{aligned} M_h V_h^n + P_h^n &= b_h^{n-1}, \\ [P_h^n]_j + \beta\, [V_h^n - TD]_j &> 0 \qquad \forall j \in \mathcal{J}^n, \\ [P_h^n]_i &= 0 \qquad \forall i \in \mathcal{I}^n, \end{aligned} \tag{51}$$

for a given positive constant β. In the above, $\mathcal{I}^n$ and $\mathcal{J}^n$ are, respectively, the *inactive* and the *active* sets at time τ_{m_n}. More precisely, the iterative algorithm builds sequences $\left\{V_{h,k}^n\right\}_k$, $\left\{P_{h,k}^n\right\}_k$, $\{\mathcal{I}_k^n\}_k$ and $\{\mathcal{J}_k^n\}_k$, converging to V_h^n, P_h^n, $\mathcal{I}^n$ and $\mathcal{J}^n$, by means of the following steps:

(1) Initialize $V_{h,0}^n = TD_h^n$ and $P_{h,0}^n = \max(b_h^n - M_h V_{h,0}^n, 0) \geq 0$. Choose $\beta > 0$. Set $k = 0$.
(2) Compute

$$\begin{aligned} Q_{h,k}^n &= \max\left\{0, P_{h,k}^n + \beta\left(V_{h,k}^n - TD_{h,k}^n\right)\right\}, \\ \mathcal{J}_k^n &= \left\{j \in \mathcal{N}, \left[Q_{h,k}^n\right]_j > 0\right\}, \\ \mathcal{I}_k^n &= \{i \in \mathcal{N}, \left[Q_{h,k}^n\right]_i = 0\}. \end{aligned}$$

(3) If $k \geq 1$ and $J_k^n = J_{k-1}^n$ then convergence is achieved. Stop.
(4) Let V and P be the solution of the linear system

$$\begin{aligned} &M_h V + P = b^{n-1}, \\ &P = 0 \text{ on } \mathcal{I}_k^n \text{ and } V = TD \text{ on } \mathcal{J}_k^n. \end{aligned} \tag{52}$$

Set $V_{h,k+1}^n = V, P_{h,k+1}^n = \max\{0, P\}, k = k + 1$ and go to 2.

It is important to note that, instead of solving the full linear system in (52), for $\mathcal{I} = \mathcal{I}_k^n$ and $\mathcal{J} = \mathcal{J}_k^n$ the following reduced one on the inactive set is solved:

$$\begin{aligned} [M_h]_{\mathcal{I}\mathcal{I}}\, [V]_{\mathcal{I}} &= \left[b^{n-1}\right]_{\mathcal{I}} - [M_h]_{\mathcal{I}\mathcal{J}}\, [TD]_{\mathcal{J}}\,, \\ [V]_{\mathcal{J}} &= [TD]_{\mathcal{J}}\,, \\ P &= b^{n-1} - M_h V. \end{aligned} \tag{53}$$

In [11], it is proved the convergence of the algorithm in a finite number of steps for a Stieltjes matrix (i.e., a real symmetric positive definite matrix with negative off-diagonal entries [18]) and a suitable initialization (the same we consider in this paper). They also proved that $\mathcal{I}_k \subset \mathcal{I}_{k+1}$. Nevertheless, a Stieltjes matrix can be only obtained for linear elements but never for the here used quadratic elements because we have some positive off-diagonal entries coming from the stiffness matrix (actually we use a lumped mass matrix). However, we have obtained good results by using ALAS algorithm with quadratic finite elements.

3.5 *Iterative method for the arbitrage free equation*

In order to obtain the interest rate which satisfies the equilibrium condition (21), a Newton method with discrete derivative (secant method) is implemented to solve $f(c) = 0$, where f is defined to balance the equilibrium condition in the form

$$f(c) = V(\tau_1 = \Delta T_1, H_{initial}, r_{initial}; \Psi, c) + I(\tau_1 = \Delta T_1, H_{initial}, r_{initial}; \Psi, c) - (1-\xi)P(0)$$

Starting from an initial value c_0 and Δ_0 the initial increment in c_0. Then the value of the mortgage components involved in the equilibrium condition are calculated with c_0. Next, we check if $f(c_0)$ is less than a given tolerance, if this condition is not satisfied we set $c_1 = c_0 + \Delta_0$ and repeat the process. At iteration i, we check if $f(c_i)$ is less than a tolerance, if it is not the case we compute

$$\Delta_i = -\frac{\Delta_{i-1} f(c_i)}{f(c_i) - f(c_{i-1})}, \quad i \geq 1 \tag{54}$$

and update $c_{i+1} = c_i + \Delta_i$ until the convergence criterium is fulfilled.

4. Numerical results

In order to obtain the solution of the fixed rate mortgage valuation problem we need to specify a set of parameters, related to the economic environment, contract characteristics and insurance. All of them, based on the literature are shown in Table 1 (see [1] and [17]). Moreover, concerning the numerical methods employed to solve the problem, we consider the parameters collected in Table 2.

Table 1. Fixed parameters in the mortgage valuation model

Economic framework	
Steady state spot rate, θ	10 %
Speed of reversion, κ	25 %
House service flow, δ	7.5%
Correlation coefficient, ρ	0
Contract specifications	
Initial value of the house, $H_{initial}$	100000€
Ratio of the loan to value	95 %
Initial estimate for contract rate, c_0	10%
Prepayment penalty, Ψ	5%
Insurance	
Guaranteed fraction of total loss, γ	80%
Cap, Γ	$20\% H_{initial}$

In Tables 3, 4, 5 and 6 the influence of different parameters (such as interest rate and house price volatilities, loan maturity, spot interest rate and arrangement fee) in the contract rate, mortgage value and insurance and coinsurance is shown.

If we increase the life of the loan the equilibrium interest rate, the insurance and coinsurance increase, however the value of the mortgage decreases as expected.

Table 2. Numerical resolution parameters

Computational domain	
H_∞	200000€
r_∞	40 %
Finite elements mesh data	
Number of elements	576
Number of nodes	2401
Time discretization	
Time steps per month	30
ALAS algorithm	
Parameter β	10000

Otherwise the effect of increasing the volatilities reduces the value of the mortgage and increases the values of the insurance and the coinsurance. This variation in the volatilities also produces and increment in the contract fixed rate.

Table 3. Contract rate, mortgage, insurance and coinsurance values for $\sigma_r = 5\%$, $\sigma_H = 5\%$ and different contract specifications

Loan (years)	spot rate r(0)	ξ	Contract rate c	Mortgage value V	Insurance I	Coinsurance CI
15	8%	0%	9.0839%	94549	449	112
		0.5%	8.9911%	94116	410	103
		1%	8.8992%	93663	386	96
		1.5%	8.8119%	93230	345	86
	10%	0%	10.0782%	94656	343	84
		0.5%	9.9696%	94208	317	79
		1%	9.8634%	93764	288	72
		1.5%	9.7579%	93316	260	66
	12%	0%	11.1662%	94691	309	76
		0.5%	11.0389%	94274	249	62
		1%	10.9203%	93870	181	45
		1.5%	10.8006%	93422	154	38
25	8%	0%	9.2605%	93961	1039	260
		0.5%	9.1876%	93549	974	243
		1%	9.1158%	93117	933	233
		1.5%	9.0453%	92677	899	225
	10%	0%	10.1258%	94314	685	171
		0.5%	10.0369%	93878	646	162
		1%	9.9440%	93417	632	158
		1.5%	9.8551%	92970	604	151
	12%	0%	11.1585%	94536	464	116
		0.5%	11.0462%	94126	399	101
		1%	10.9219%	93667	382	94
		1.5%	10.8111%	93240	337	85

Figures 1 to 3 illustrate the values at origination of the contract, insurance and coinsurance when the arrangement fee is equal to 0.5% and the early prepayment penalty takes the value of 5%. We consider the fixed parameters of the model shown in Table 1. In this case the contract rate is 9.3969%, the interest rate volatility is 10%, the house price volatility is 5%, the maturity of the contract is 25 years and the spot rate is 8%. Moreover, Figure 4 shows the prepayment (coincidence) region in red and the non early prepayment (non coincidence) region in blue, the curve separating both regions is the optimal prepayment boundary (free boundary). The prepayment region coincides with high house prices and low interest rates because default is unlikely at high house values so the borrower is willing to prepay at high interest rates.

Finally, Table 7 shows the results for a case with higher volatility in the house price (20%). We notice that as soon as volatility becomes higher, although it results much cheaper from the computational point of view, neglecting second order terms in the PDE as proposed with the perturbation method in [17] can produce very inaccurate prices. On the other hand, the increase in volatility produces a decrease in the mortgage value and an increase in the insurance as expected.

Table 4. Contract rate, mortgage, insurance and coinsurance values for $\sigma_r = 10\%$, $\sigma_H = 5\%$ and different contract specifications

Loan(years)	spot rate r(0)	ξ	Contract rate c	Mortgage value V	Insurance I	Coinsurance CI
15	8%	0%	9.3028%	94390	609	152
		0.5%	9.1741%	93959	566	141
		1%	9.0484%	93523	526	131
		1.5%	8.9184%	93064	511	128
	10%	0%	10.5172%	94506	494	124
		0.5%	10.3544%	94065	459	115
		1%	10.1925%	93621	429	107
		1.5%	10.0424%	93196	378	95
	12%	0%	11.8193%	94610	389	97
		0.5%	11.6207%	94161	364	91
		1%	11.4324%	93723	327	81
		1.5%	11.2617%	93270	305	76
25	8%	0%	9.5142%	93778	1222	306
		0.5%	9.3969%	93315	1209	302
		1%	9.2833%	92847	1202	300
		1.5%	9.1746%	92387	1187	296
	10%	0%	10.6232%	94102	898	224
		0.5%	10.4877%	93688	836	209
		1%	10.3441%	93235	815	203
		1.5%	10.2052%	92780	795	198
	12%	0%	11.8641%	94344	655	163
		0.5%	11.6778%	93885	639	159
		1%	11.4993%	93430	620	155
		1.5%	11.3534%	93017	557	138

Table 5. Contract rate, mortgage, insurance and coinsurance values for $\sigma_r = 5\%$, $\sigma_H = 10\%$ and different contract specifications

Loan (years)	spot rate r(0)	ξ	Contract rate c	Mortgage value V	Insurance I	Coinsurance CI
15	8%	0%	9.0078%	92650	2350	587
		0.5%	8.9084%	92242	2282	571
		1%	8.8132%	91845	2205	551
		1.5%	8.7195%	91446	2129	532
	10%	0%	10.0154%	92984	2015	503
		0.5%	9.8983%	92565	1960	490
		1%	9.7861%	92154	1896	474
		1.5%	9.6801%	91748	1826	456
	12%	0%	11.1181%	93270	1730	432
		0.5%	10.9775%	92849	1676	418
		1%	10.8459%	92427	1622	405
		1.5%	10.7241%	92015	1559	389
25	8%	0%	9.2191%	91407	3594	898
		0.5%	9.1386%	90991	3533	882
		1%	9.0585%	90565	3484	870
		1.5%	8.9818%	90144	3430	857
	10%	0%	10.0815%	91997	3003	751
		0.5%	9.9881%	91590	2934	733
		1%	9.9022%	91204	2845	711
		1.5%	9.8104%	90778	2797	699
	12%	0%	11.1048%	92532	2468	624
		0.5%	10.9742%	92090	2434	608
		1%	10.8564%	91675	2376	592
		1.5%	10.7423%	91239	2235	584

5. Conclusions

In this paper we first revise the statement of the PDE model for pricing fixed rate mortgages with prepayment and default options. Next, a set of numerical techniques for solving the associated to problems to obtain mortgage, insurance and coinsurance values, as well as the optimal prepayment boundary (free boundary). Taking into account the convection dominated feature of the PDE, specially in the case of low volatilities, a time discretization based on the upwinding characteristics Crank-Nicolson scheme is proposed and combined with finite element methods. For the free boundary mortgage pricing problem associated to prepayment option, a suitable augmented Lagrangian algorithm is proposed. The equilibrium interest

Table 6. Contract rate, mortgage, insurance and coinsurance values for $\sigma_r = 10\%$, $\sigma_H = 10\%$ and different contract specifications

Loan(years)	spot rate r(0)	ξ	Contract rate c	Mortgage value V	Insurance I	Coinsurance CI
15	8%	0%	9.2331%	92591	2409	602
		0.5%	9.1019%	92182	2343	586
		1%	8.9759%	91779	2271	568
		1.5%	8.8473%	91358	2217	554
	10%	0%	10.4358%	92933	2066	517
		0.5%	10.2713%	92508	2017	504
		1%	10.1134%	92086	1963	490
		1.5%	9.9619%	91662	1914	479
	12%	0%	11.7276%	93237	1762	440
		0.5%	11.5309%	92801	1724	431
		1%	11.3515%	92376	1674	418
		1.5%	11.1841%	91943	1632	408
25	8%	0%	9.4344%	91298	3701	926
		0.5%	9.3221%	90862	3663	917
		1%	9.2165%	90434	3615	906
		1.5%	9.1125%	90001	3574	896
	10%	0%	10.5161%	91935	3065	766
		0.5%	10.3746%	91492	3033	758
		1%	10.2381%	91049	3001	750
		1.5%	10.1078%	90608	2966	740
	12%	0%	11.7368%	92498	2502	626
		0.5%	11.5608%	92048	2476	619
		1%	11.3896%	91582	2468	616
		1.5%	11.2423%	91149	2426	606

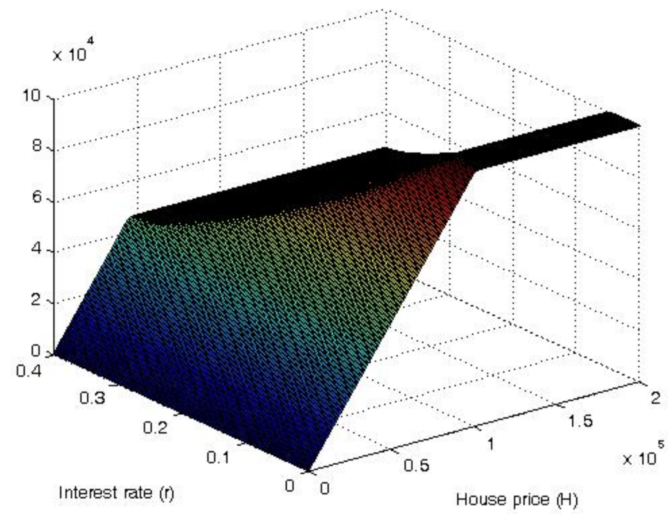

Figure 1. Mortgage value at origination

Table 7. Contract rate, mortgage, insurance and coinsurance values for $\sigma_r = 10\%$, $\sigma_H = 20\%$ and different contract specifications

Loan(years)	spot rate r(0)	ξ	Contract rate c	Mortgage value V	Insurance I	Coinsurance CI
15	8%	0%	9.3117%	87941	7059	2036
		0.5%	9.1721%	87538	6987	2050
		1%	9.0397%	87150	6900	2068
		1.5%	8.9103%	86760	6815	2078
	10%	0%	10.4659%	88460	6540	1780
		0.5%	10.3083%	88068	6457	1788
		1%	10.1506%	87664	6386	1790
		1.5%	10.0025%	87270	6305	1799
	12%	0%	11.7052%	88964	6036	1591
		0.5%	11.5177%	88550	5975	1587
		1%	11.3434%	88146	5904	1585
		1.5%	11.1749%	87738	5837	1584

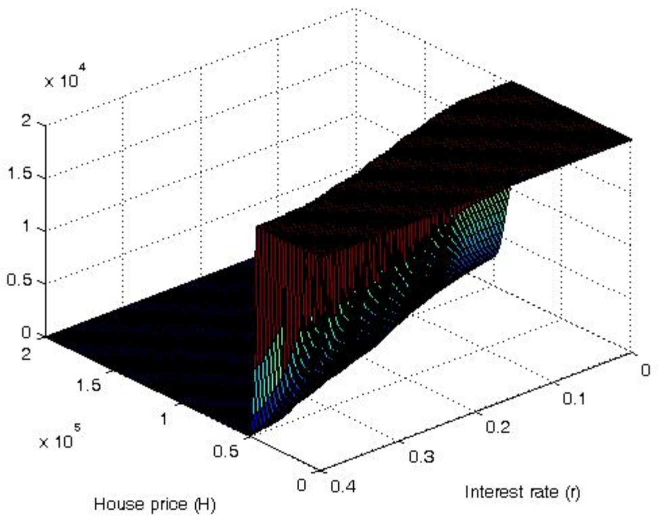

Figure 2. Insurance value at origination

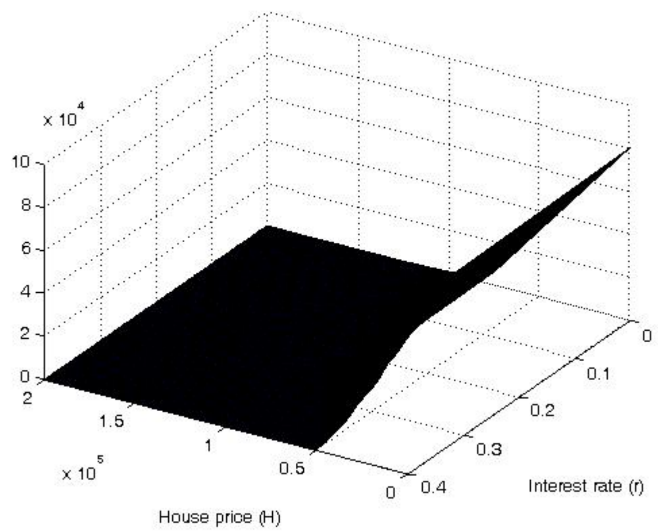

Figure 3. Coinsurance value at origination

rate of the loan is obtained by a Newton method. Numerical techniques are differeent from those ones proposed in [17]. Numerical results illustrate the performance of the proposed numerical techniques and show the expected qualitative behaviour of the mortgage, insurance and coinsurance values, as well as the optimal prepayment boundary that separates the prepayment and non prepayment regions. The proposed set of numerical techniques is also suitable for the case of larger volatilities, where the use of perturbation techniques would require the consideration of higher order terms in the asymptotic expansion, thus increasing the complexity of the model equations and the computational cost. This is deeply analyzed in [19] for the vanilla European and American options setting.

As future work, the authors aim the consideration of jumps in the stochastic process for the house prices by means of a jump-diffusion model, which would lead to a partial integro-differential equation (PIDE) and perhaps better reflects the evolution of real state prices in the financial crisis setting in many countries.

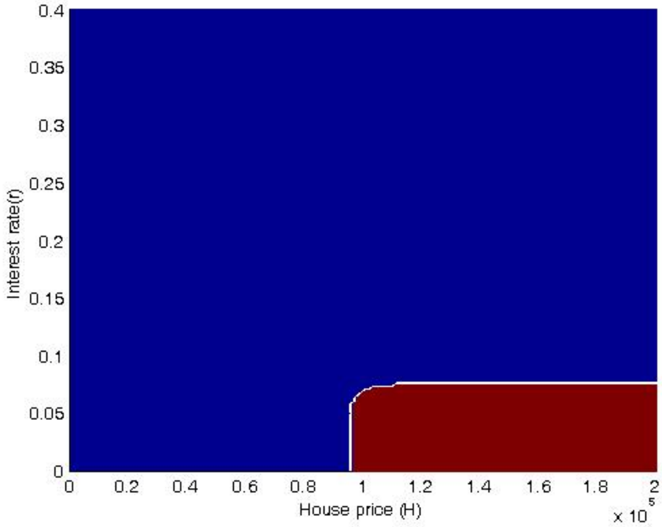

Figure 4. Free boundary at origination

References

[1] J.A. Azevedo-Pereira, D.P. Newton & D.A. Paxson, *UK Fixed Rate Repayment Mortgage and Mortgage Indemnity Valuation*, Real Estate Economics 30 (2002), pp. 185-211.

[2] A. Bermúdez, M.R. Nogueiras & C. Vázquez, *Numerical analysis of convection-diffusion-reaction problems with higher order characteristics finite elements. Part I: Time discretization*, SIAM J. Numer. Anal. 44 (2006), pp. 1829-1853.

[3] A. Bermúdez, M.R. Nogueiras & C. Vázquez, *Numerical analysis of convection-diffusion-reaction problems with higher order characteristics finite elements. Part II: Fully discretized scheme and quadrature formulas*, SIAM J. Numer. Anal. 44 (2006), pp. 1854-1876.

[4] A. Bermúdez, M.R. Nogueiras & C. Vázquez, *Numerical solution of variational inequalities for pricing Asian options by higher order Lagrange-Galerkin methods.* Applied Numerical Mathematics 56 (2006), pp. 1256-1270.

[5] M.C. Calvo-Garrido & C. Vázquez, *Pricing pension plans based on average salary without early retirement: PDE modeling and numerical solution*, J. of Computational Finance 16 (2012), pp. 111-140.

[6] M.C. Calvo-Garrido, A. Pascucci & C. Vázquez, *Mathematical analysis and numerical methods for pricing pension plans allowing early retirement*, SIAM J. of Applied Mathematics 73 (2013), pp. 1747–1767.

[7] J.C. Cox, J.E. Ingersoll & S.A. Ross, *A theory of the term structure of interest rates*, Econometrica 53 (1985), pp. 385-407.

[8] J.E. Hilliard, J.B. Kau & V.C. Slawson, *Valuing Prepayment and Default in a Fixed-Rate Mortgage: A Bivariate Binomial Options Pricing Technique*, Real Estate Economics, 26 (1998), pp. 431-468.

[9] K. Itô, *On Stochastic Differential Equations*, Memoirs American Mathematical Society, 4 (1951), pp. 1-51.

[10] R. Kangro & R. Nicolaides, *Far field boundary conditions for Black-Scholes equations*, SIAM J. on Numerical Analysis 38 (2000), pp. 1357-1368.

[11] T. Kärkkäinen, K. Kunisch & P. Tarvainen, *Augmented Lagrangian Active Set methods for obstacle problems*, J. Optim. Theory Appl. 19 (2003), pp. 499-533.

[12] J. Kau, D.C. Keenan, W.J. Muller & J.F. Epperson, *The Valuation at Origination of Fixed-Rate Mortgages with Default and Prepayment*, Journal of Real Estate Finance and Economics 11 (1995), pp. 5-36

[13] L. Jiang, B. Bian & F. Yi *A parabolic variational inequality arising from the valuation of fixed rate mortgages*, European Journal of Applied Mathematics 16 (2005), pp. 361-383

[14] R.C. Merton, R. C. *The theory of rational option pricing*, Bell J. of Economics and Management Science 4 (1973), pp. 141-183.

[15] O.A. Oleinik & E.V. Radkevic, *Second Order Equations with Nonnegative Characteristic Form*, American Mathematical Society, Providence, Rl, 1973.

[16] N.J. Sharp, *Advances in Mortgage Valuation: an Option-Theoretic Approach*, PhD thesis: School of Mathematics, The University of Manchester, 2006.

[17] N.J. Sharp, D.P. Newton & P.W. Duck *An Improved Fixed-Rate Mortgage Valuation Methodology with Interacting Prepayment and Default Options*, J. of Real Estate Finance and Economics 19 (2008), pp. 49-67.

[18] R.S. Varga, *Matrix iterative analysis*, Prentice-Hall Inc., Englewood Cliffs, N.J., 1962.

[19] M. Widdicks, P.W. Duck, D. Andricopoulos & D.P. Newton *The Black-Scholes equation revisited: asymptotic expansions and singular perturbations*, J. of Mathematical Finance 15 (2005), pp. 373-392.

[20] P. Wilmott, J.N. Dewynne & S. Howison, *Option Pricing: Mathematical Models and Computation*, Oxford Financial Press, Oxford, UK, 1993.